OPEN ACCESS

Kashf Journal of Multidisciplinary Research
Vol: 03 - Issue 05 (2026)
P-ISSN: 3007-1992 E-ISSN: 3007-200X
https://kjmr.com.pk

NON-CLASSICAL LIE SYMMETRY ANALYSIS OF KDV, JAULENT–MIODEK, AND JIMBO–MIWA EQUATIONS

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DOI:(*https://doi.org/10.71146/kjmr917*)

Article Info

Abstract

The Lie symmetry analysis of non-classical is presented in the context of three important nonlinear partial differential equations, the (1+1)-dimensional Kortewegde Vries (KdV) equation, the (1+2)-dimensional Jaulent-Miodek (JM) equation and the (1+3)-dimensional modified Jimbo-Miwa equation. The non-classical symmetry approach, a variation on the classical Lie symmetry analysis, by adding an invariant surface condition, is employed to achieve symmetry reductions and invariant solutions of these equations. The corresponding determining equations are obtained and solved to construct non-classical symmetry generators. These generators are subsequently used to elicit smaller forms and even exact solutions to the invariance.

The results of the analysis are subsequently validated and interpreted by visualizing representative solutions through contour plots, surface plots and line profiles. Graphical Analysis the KdV equation forms localised travelling wave structures, the JM equation gives periodic wave structures in two space dimensions and the modified Jimbo-Miwa equation, demonstrates complex multidimensional interactions of waves. The findings underscore how the non-classical symmetry approach can be efficiently used to find physically significant solutions, especially to higher-dimensional nonlinear PDEs. The study extends the already existing literature on non-classical symmetry analysis to the multidimensional models and provides both analytical and graphical understanding of the behaviour of the solutions involving the multidimensional models.

Keywords: *Non-classical Lie symmetry; invariant solutions; KdV equation; Jaulent–Miodek equation; Jimbo–Miwa equation; symmetry reduction*

1. Introduction

Partial differential equations (PDEs) are equations that are used in fundamental physics, engineering, and applied science to model a diverse array of physical, engineering, and biological phenomena, such as fluid dynamics, nonlinear wave propagation, plasma physics, and mathematical biology. These equations are the interrelations between the functions of many variables and their partial derivatives, and their solutions can usually give insight into the behaviour of complex systems [1], [2]. Nonlinear PDEs are however typically hard to solve analytically and, as such, the construction of systematic schemes to obtain exact or approximate solutions is a significant field of research [3].

Lie symmetry analysis is one of the most effective and popular methods of analysing differential equations and was first introduced by Sophus Lie at the end of the nineteenth century. Lies theory offers a systematic approach to the discovery of continuous symmetry transformations that fix a given differential equation [4]. They can be applied to the reduction of the order or dimensionality of differential equations to get a set of solutions that are invariant, similarity reductions and exact analytical forms. In the last decades, Lie symmetry methods are widely developed and applied to a range of both linear and nonlinear differential equations [5], [6].

Although effective, classical Lie symmetry analysis is not always able to produce a complete set of solutions, especially of nonlinear PDEs. To address this shortcoming, Bluman and Cole introduced non-classical symmetry analysis or conditional symmetry analysis. The approach is a generalization of the classical method in that it uses conditions of invariant surfaces in the symmetry analysis and, therefore, permits the construction of other symmetries and classes of solutions that are inaccessible by classical methods [5], [6]. The non-classical approach has shown to be an effective means of finding similarity reductions and exact solutions of nonlinear PDEs, particularly in higher dimensions.

Non-classical symmetry methods have in recent years been effectively used in solving various significant nonlinear evolution equations, such as the KdV equation and generalizations to higher dimensions. The KdV equation is a famous model that explains nonlinear wave propagation in shallow water and other physical systems and the Jimbo-Mioka equations and Jimbo-Miowa equations are a consequence of studying integrable systems and multidimensional nonlinear wave phenomena [7]. The equations are significant test cases in the design of symmetry-based solution methodologies because of their strong mathematical structure and physical applicability.

The current paper is devoted to **non-classical Lie symmetry analysis** applied to the three important nonlinear PDEs, namely the $(1 + 1)$-dimensional generalized KdV equation, $(1 + 2)$-dimensional JM equation, and $(1 + 3)$-dimensional modified Jimbo-Miwa equation. Using the non-classical symmetry approach, conditions of the invariant surface are built, and determining equations are obtained. These equations are then solved to get new non-classical symmetries and then these are further used to get similarity reductions and exact invariant solutions.

The primary value of the work is the systematic building of non-classical symmetries of these nonlinear PDEs in higher dimensions and the derivation of invariant solutions. The findings indicate that the non-classical symmetry approach is effective in increasing the scope of the solvable nonlinear models and give further information on the structure of multidimensional nonlinear evolution equations. The organization of the article is as follows. The section 2 is earmarked for literature review whereas the section 3 presents Non-Classical Symmetry Analysis and Graphical Discussion of Invariant Solutions. At the end, conclusion is given in section 4.

2. Literature Review

This use of symmetry techniques on differential equations has been a major subject of study in applied mathematics since the original work of Sophus Lie who proposed continuous transformation groups as a work system in solving differential equations. Lie symmetry analysis contribute as a strong tool to simplify differential equations and formulate invariant solutions and has been widely developed both in concept and in computer codes [8], [5]. Further developments have shown that symmetry-based methods are especially useful in the case of nonlinear partial differential equations, where classical methods of analysis can prove ineffective.

Similarity reductions and exact solutions of nonlinear PDEs have been commonly obtained by classical Lie symmetry methods. As an example, Olver and Rosenau came up with systematic methods of constructing group-invariant solutions and special exact solutions of differential equations [6] [9] [10]. With these techniques, other nonlinear evolution equations such as diffusion equations and wave propagation models have been successfully solved [11]. It has however been noted that classical methods of symmetry are not always applicable to find all reductions as well as solutions, particularly with complex nonlinear systems.

To overcome these restrictions, Bluman and Cole introduced non-classical symmetry analysis (or conditional symmetry analysis). This method is an extension of classical Lie symmetry methods with the added condition of the invariant surface, resulting in further symmetry reductions which cannot be achieved via classical methods [5]. The equations resulting of the non-classical symmetries are more complex and nonlinear and allow the fabrication of other richer structures of solutions. More recently, direct reduction techniques and differential constraint techniques have been developed that have broadened the application of symmetry techniques to a wider range of nonlinear PDEs [6] [12].

Various important non-linear PDEs in mathematical physics have been studied using alternative approaches based on generalized and non-classical symmetry. The KdV equation has been of much interest because it has been important to study nonlinear waves. In a variety of studies symmetry analysis and direct methods have been used to derive similarity reductions and even exact solutions of KdV-type equations, in higher-dimensional generalizations. [7]. These papers emphasize the practicality of symmetry-based methods to the nonlinear dynamics of waves.

Similarly, nonlinear PDEs of higher dimensions like Jaulent-Miodek equations and Jimbo Miwa equations have also been explored as integrable systems and nonlinear evolution equations. These models are developed in the analysis of multidimensional wave interactions, and have rich mathematical structures. Symmetry analysis of such equations has resulted in the new invariant solutions, conservation laws, and reduction methods. Nevertheless, the majority of the available literature is devoted to classical symmetry techniques, or direct reduction techniques, and there are relatively fewer papers on non-classical symmetries in higher dimensions.

The recent trends have also given prominence to the computational tools and symbolic software in determining symmetries and invariant solutions [4] [13]. These developments have led to the systematic study of complex nonlinear systems, and have helped researchers deal with the complex determining equations of non-classical symmetries.

Although these have important contributions, it is still necessary to further investigate non-classical symmetry analysis of higher-dimensional nonlinear PDEs, especially in equations like the JM equation and the modified Jimbo-Miwa equation. The new non-classical symmetries and their associated invariant solutions can give more information about the structure of these equations and widen the scope of models that are analytically solvable.

In this respect, the current research paper adds to the current literature through the application of the non-classical Lie symmetry analysis to the (1+1)-dimensional generalized KdV equation, the (1+2)-dimensional JM equation, and the (1+3)-dimensional modified Jimbo-Miwa equation. The paper is concerned with the derivation of determining equations given invariant surface conditions, creation of new symmetry generators, and achieving similarity reductions and exact solutions. This methodology does not only provide a complement to the available classical symmetry results but also indicates the usefulness of non-classical methods to provide new solution structures of multidimensional nonlinear PDEs.

3. Non-Classical Symmetry Analysis and Graphical Discussion of Invariant Solutions

3.1 Korteweg–de Vries Equation

The $(1+1)$-dimensional KdV equation is given by

$$u_t + a\,u\,u_x + u_{xxx} = 0 \tag{1}$$

With the non-classical symmetry, the infinitesimal generator is

$$W = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \tag{2}$$

with invariant surface condition

$$\xi^1 u_t + \xi^2 u_x - \eta = 0 \tag{3}$$

After reduction, the invariant solutions are obtained as

$$u\left(t,x\right)=-c_3t+F\left(x\right) \tag{4}$$

which represents a linearly time-dependent invariant solution with arbitrary spatial structure. Another reduced form is

$$u\left(t,x\right)=F\left(x\right) \tag{5}$$

which corresponds to a stationary or time-independent wave profile. A further invariant solution is obtained as

$$u\left(t,x\right)=F\left(x\right)e^{(-at)} \tag{6}$$

which describes an exponentially decaying wave solution due to the damping factor e^{-at}.

For graphical representation, consider the solution

$$u(t,x)=t+sech^2(x-t) \tag{7}$$

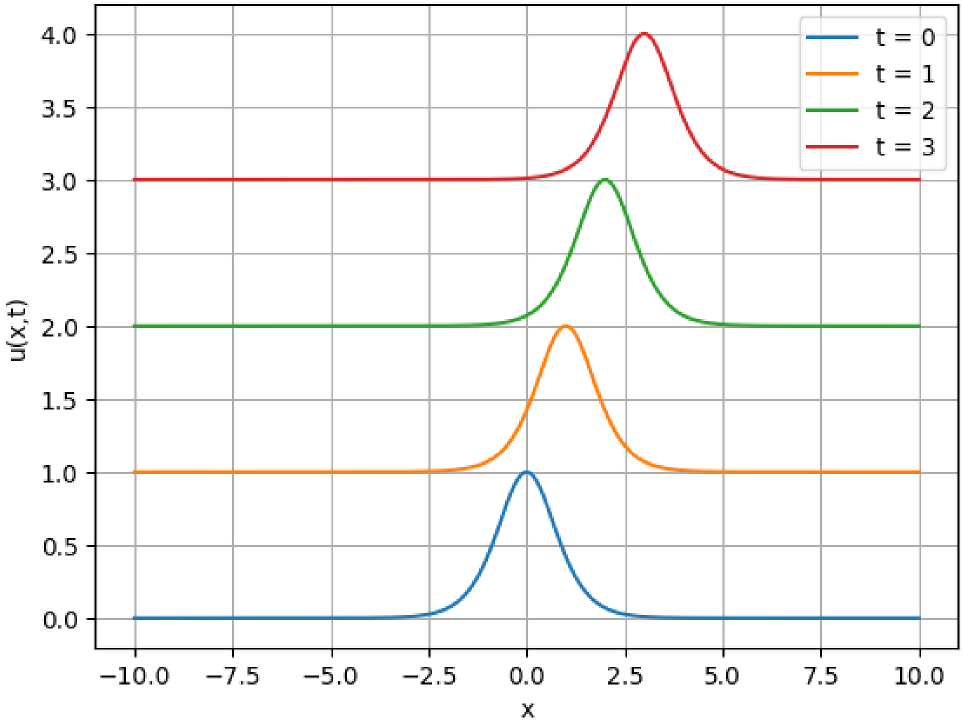

Figure 1: Travelling wave profile of KdV solution

Figure 1 depicts the travelling wave profile in which one can observe the localized peak which propagates along the spatial direction as time progresses. The wave does not change its form but only changes its location which is one of the main peculiarities of nonlinear dispersive systems. The additive term t is what causes the amplitude to increase in a linear manner over time, whereas the localized structure is maintained by the sech ^{2} component. This proves that the solution of the invariant form obtained is a stable travelling wave, which is in line with the physical behaviour of the KdV equation.

3.2 Jaulent–Miodek Equation

The (1+2)-dimensional JM equation is given by

$$au_x t - u_{xxxx} + b(u_x)^2 u_{xx} - cu_{xx}u_y - du_x u_{xy} + eu_{yy} = 0 \tag{8}$$

The infinitesimal generator is

$$W = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} \tag{9}$$

with invariant condition

$$\xi^1 u_t + \xi^2 u_x + \xi^3 u_y - \eta = 0 \tag{10}$$

The derived invariant solutions include

$$u(t,x,y) = tx + F(t,y) \tag{11}$$

which represents a coupled time-space invariant structure with linear dependence on x and t.

Another invariant solution is

$$u(t,x,y) = \frac{1}{2}t^2 + F(x-ct,y) \tag{12}$$

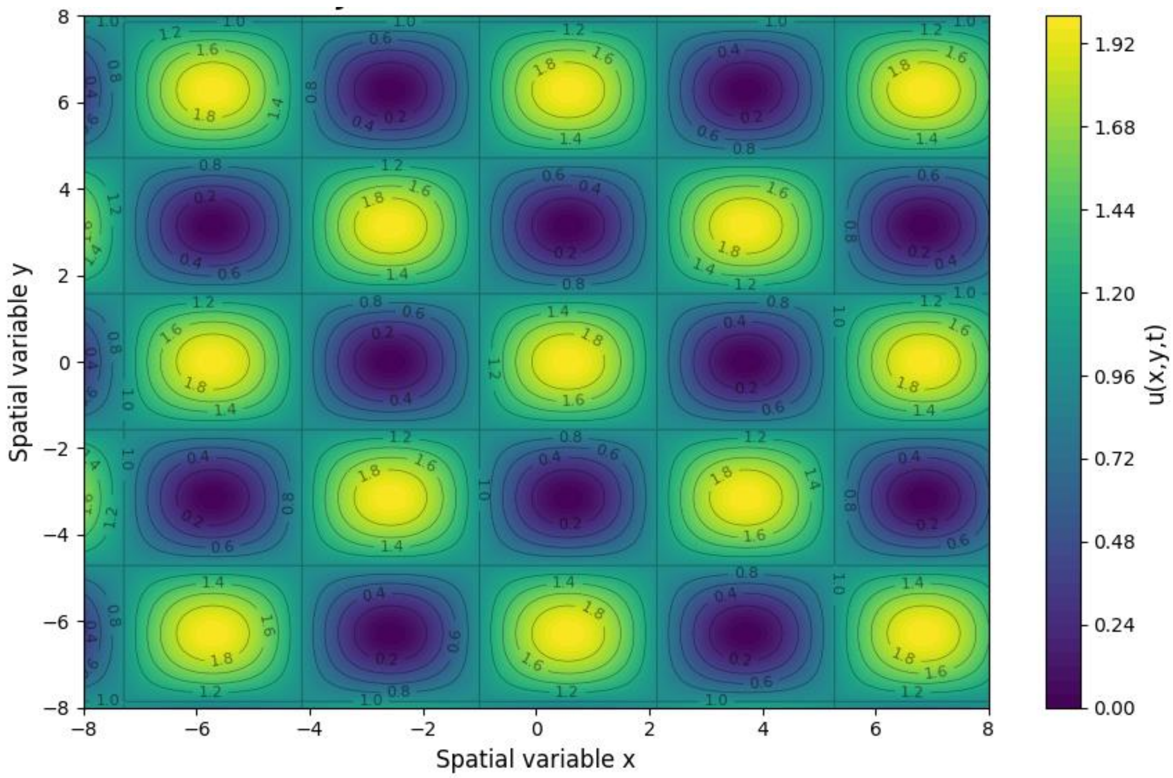

Figure 2: Contour plot of JM solution at $t = 1$

which corresponds to a travelling-wave-type solution propagating along the x-direction.

A further reduced solution is obtained as

$$u(t,x,y)=xy+F(t) \tag{13}$$

which describes nonlinear spatial coupling between the variables $x$and $y$with independent temporal evolution.

For visualization, the representative solution

$$u(t,x,y)=t+\sin(x+t)\cos(y) \tag{14}$$

is selected by choosing an appropriate form of the arbitrary function $F$to study the qualitative behavior of the invariant solutions.

Periodic structures are observed in both spatial directions in the contour plot in Figure 2. The repetitive peaks and saddles imply the solution has a regular oscillatory behaviour, that is, the nonlinear interplay of x and y.

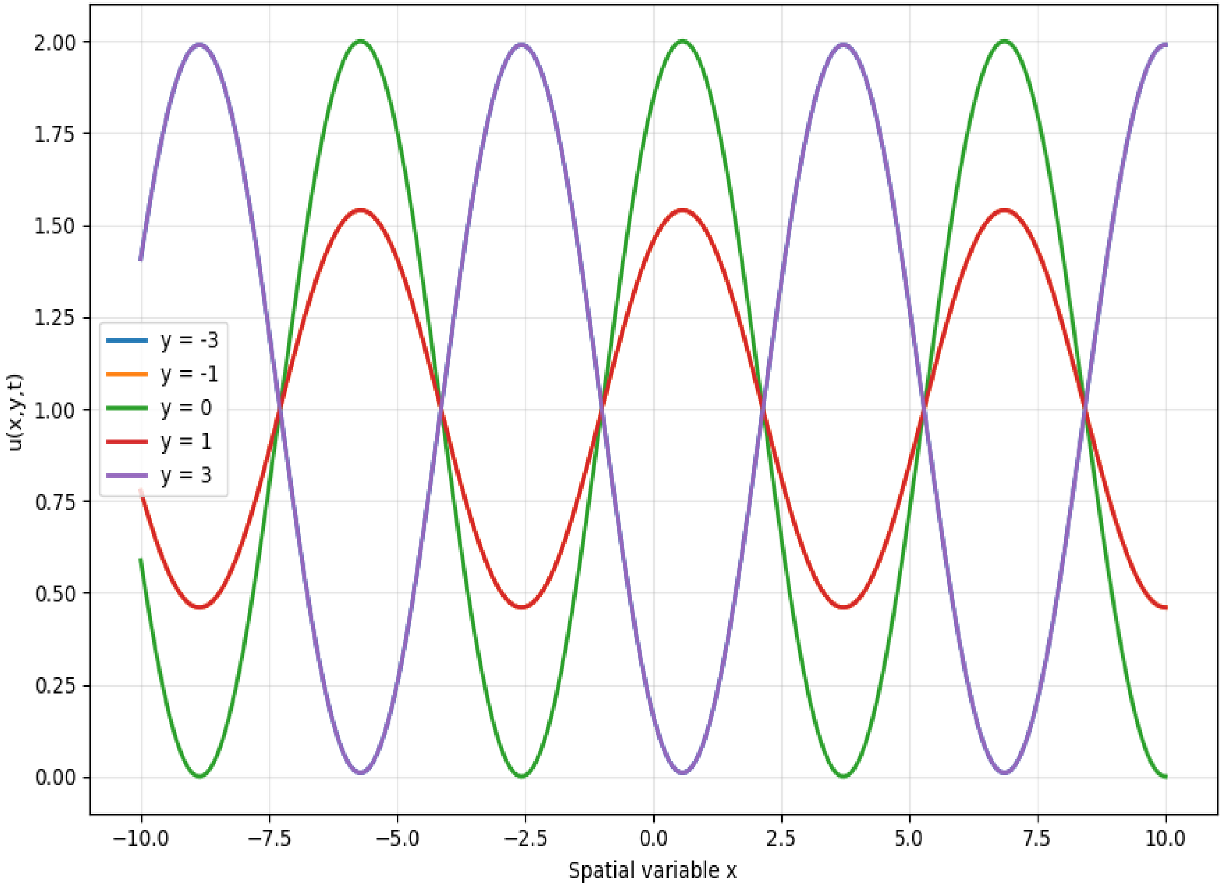

Figure 3: Cross-sectional profiles of JM solution

The cross-sectional profiles of various fixed values of y are shown in Figure 3. The curves exhibit sinusoidal behaviour with respect to the x-direction, and distinct phase differences between the two curves.

This proves that the solution is spatially dependent and waves travel along the x-axis.

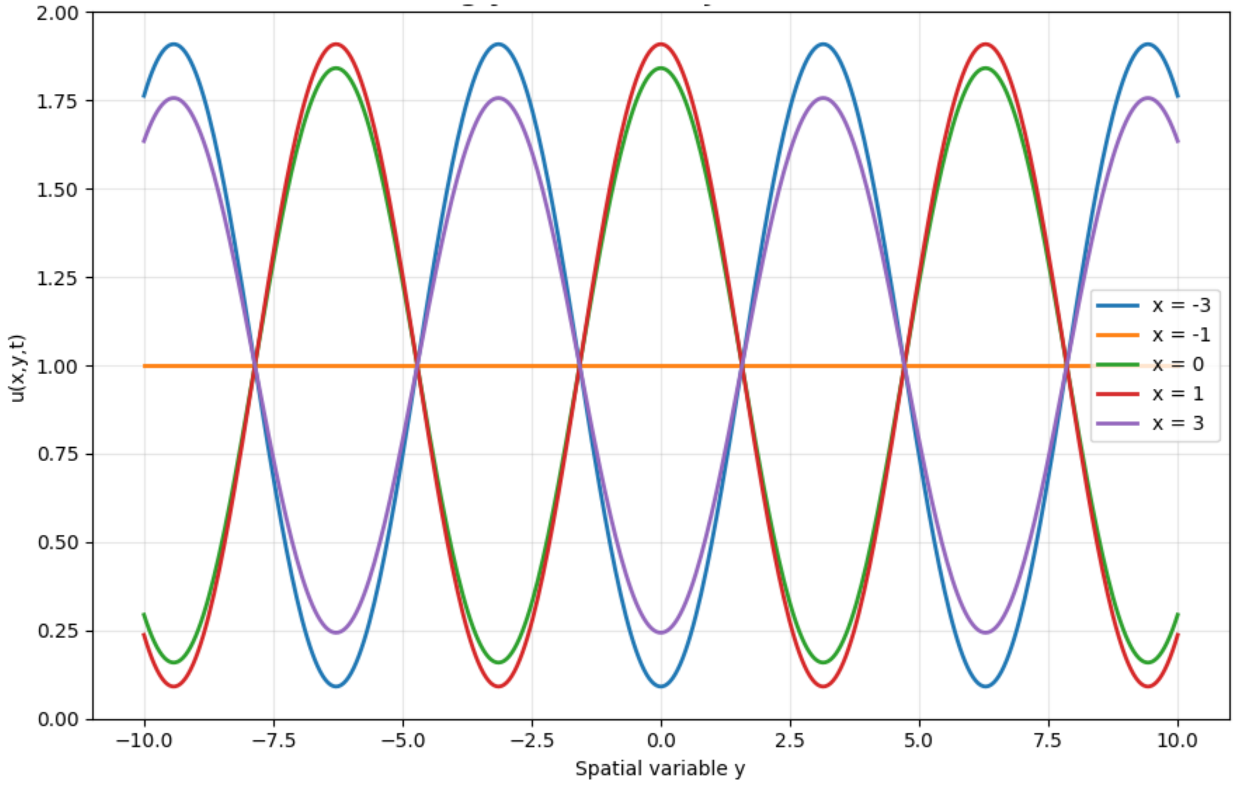

Figure 4: Variation along y-direction of JM solution

The change of the solution with respect to the y-direction as depicted in Figure 4 indicates that the solution also varies periodically in the second spatial direction denoted by y.

One of the curves looks almost constant, which is the steady-state part, whereas the other curves follow oscillatory patterns. This brings out anisotropic properties of the solution.

A three-dimensional view of the solution can be obtained by looking at the surface provided in left side of plot 5. The wave-like surface exhibits smooth peaks and troughs in a grid pattern, which confirms the coupling between spatial variables although the right side of Figure 5 shows the evolution of the surface later. The fact that the amplitude increases with time can be interpreted to mean that the solution changes over time but still retains its periodic structure. This is an indication of nonlinear dynamic behaviour of the system.

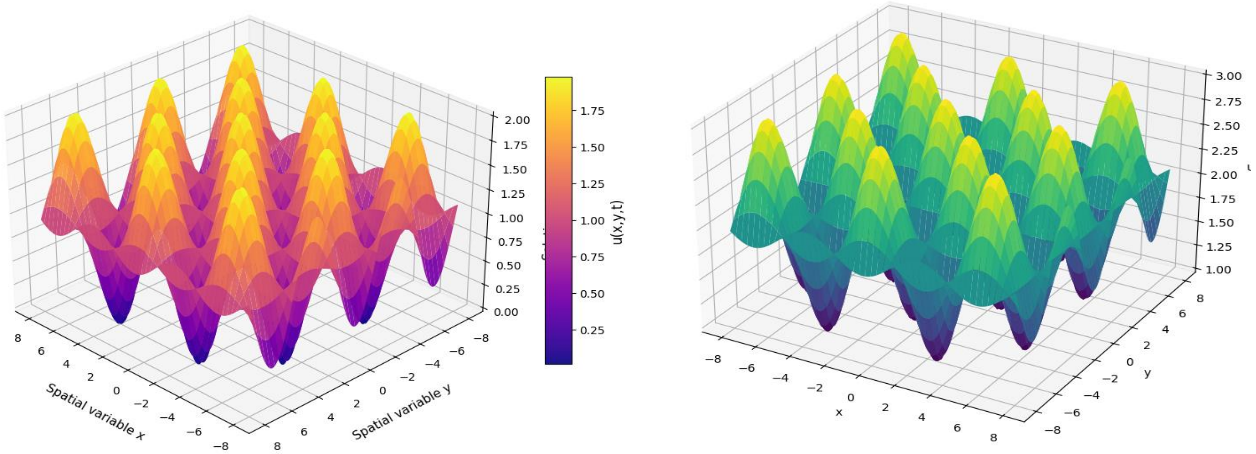

Figure 5: 3D surface plot of JM equation at t = 1 and t = 2 from left to right

3.3 Modified Jimbo–Miwa Equation

The extended (1+3)-dimensional modified Jimbo–Miwa equation is

$$au_{yxxx}+bu_xu_{xx}+cu_xu_{yx}+d\,u_{yt}+eu_{zx}=0 \tag{15}$$

The infinitesimal generator is

$$W=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\xi^4\frac{\partial}{\partial z}+\eta\frac{\partial}{\partial u} \tag{16}$$

The invariant surface condition is

$$\xi^1u_t+\xi^2u_x+\xi^3u_y+\xi^4u_z-\eta=0 \tag{17}$$

The derived invariant solutions include

$$u(t,x,y,z)=\frac{a}{b}xt+F\left(x,z,y-\frac{c}{b}zt\right) \tag{18}$$

which represents a multidimensional travelling-wave-type solution involving coupling between the spatial variables and temporal evolution. Another reduced invariant solution is

$$u(t,x,y,z)=tx+F(t,y,z) \tag{19}$$

which describes a linear interaction between the temporal and spatial variables together with an arbitrary multidimensional wave structure. A further invariant solution is obtained as

$$u(t,x,y,z)=xy+F(t) \tag{20}$$

which shows nonlinear coupling between the spatial variables $x$and $y$with independent time dependence. Similarly,

$$u(t,x,y,z) = xz + F(t) \tag{21}$$

represents interaction between the variables $z$and x, indicating the effect of the additional spatial dimension on the wave dynamics. For graphical analysis, the representative solution

$$u(t,x,y,z) = t + \sin(x+y) + \cos(z) \tag{22}$$

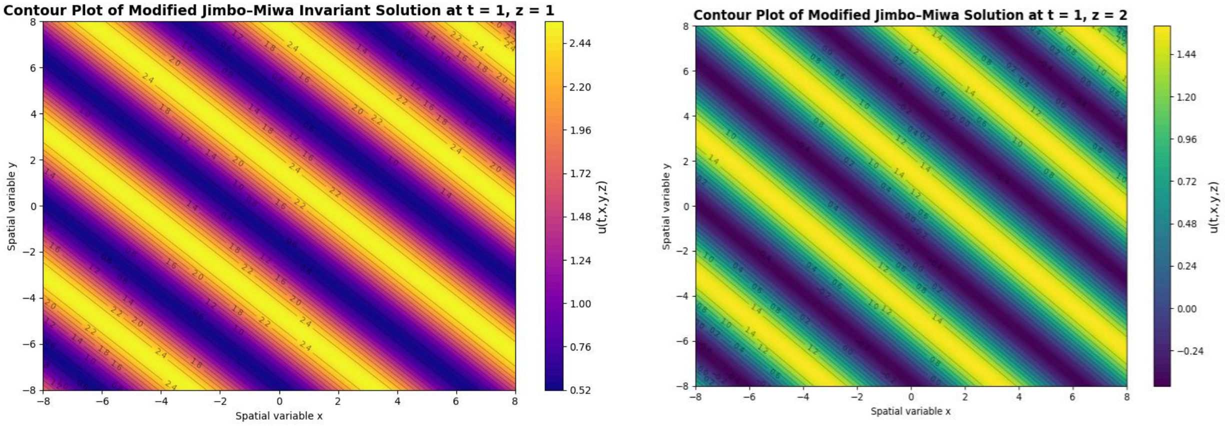

Figure 6: Contour plot of Modified Jimbo-Miwa Invariant at t = 1, z = 1 and t = 1, z = 2

is considered by selecting an appropriate form of the arbitrary function $F$to study the qualitative behaviour of the invariant solutions.

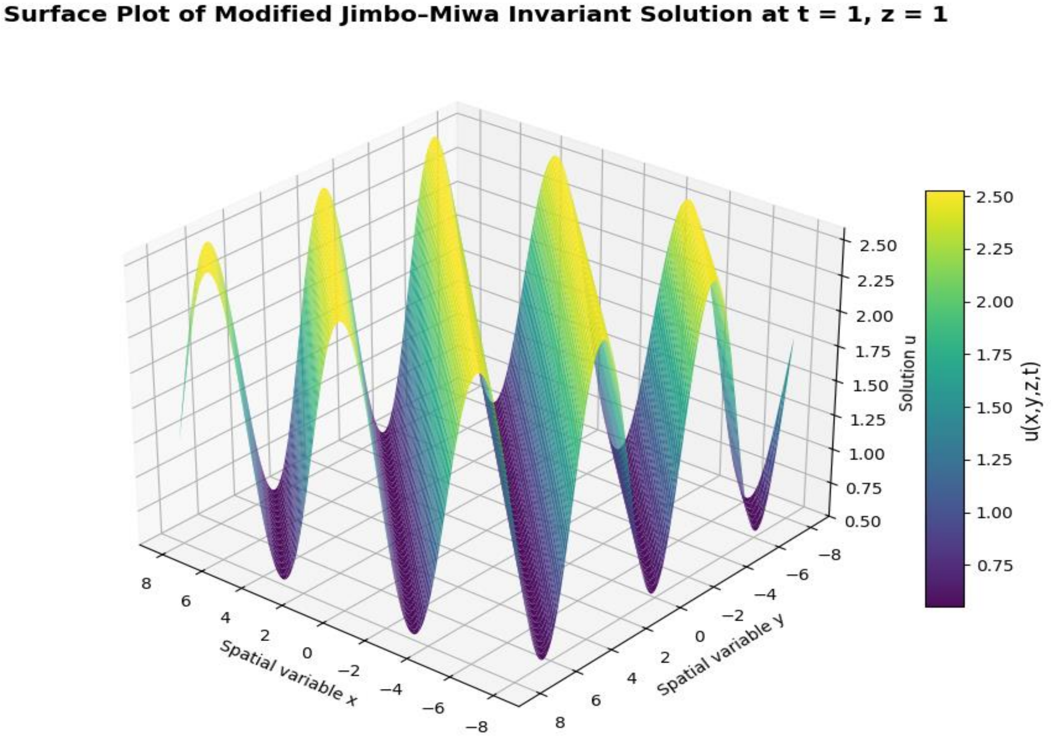

Figure 7: Surface Plot of Modified Jimbo-Miwa Invariant Solution at t = 1, z = 1

plots 6 show diagonal wave patterns, which reflects the behaviour of travelling waves in more than a single spatial direction. The parameter z changes the amplitude and orientation of the contours and shows the effect of the third spatial dimension.

Surface plot 7 reveals wave ridges that are propagating at a diverted position. This validates the existence of multi-directional interaction of waves in the system.

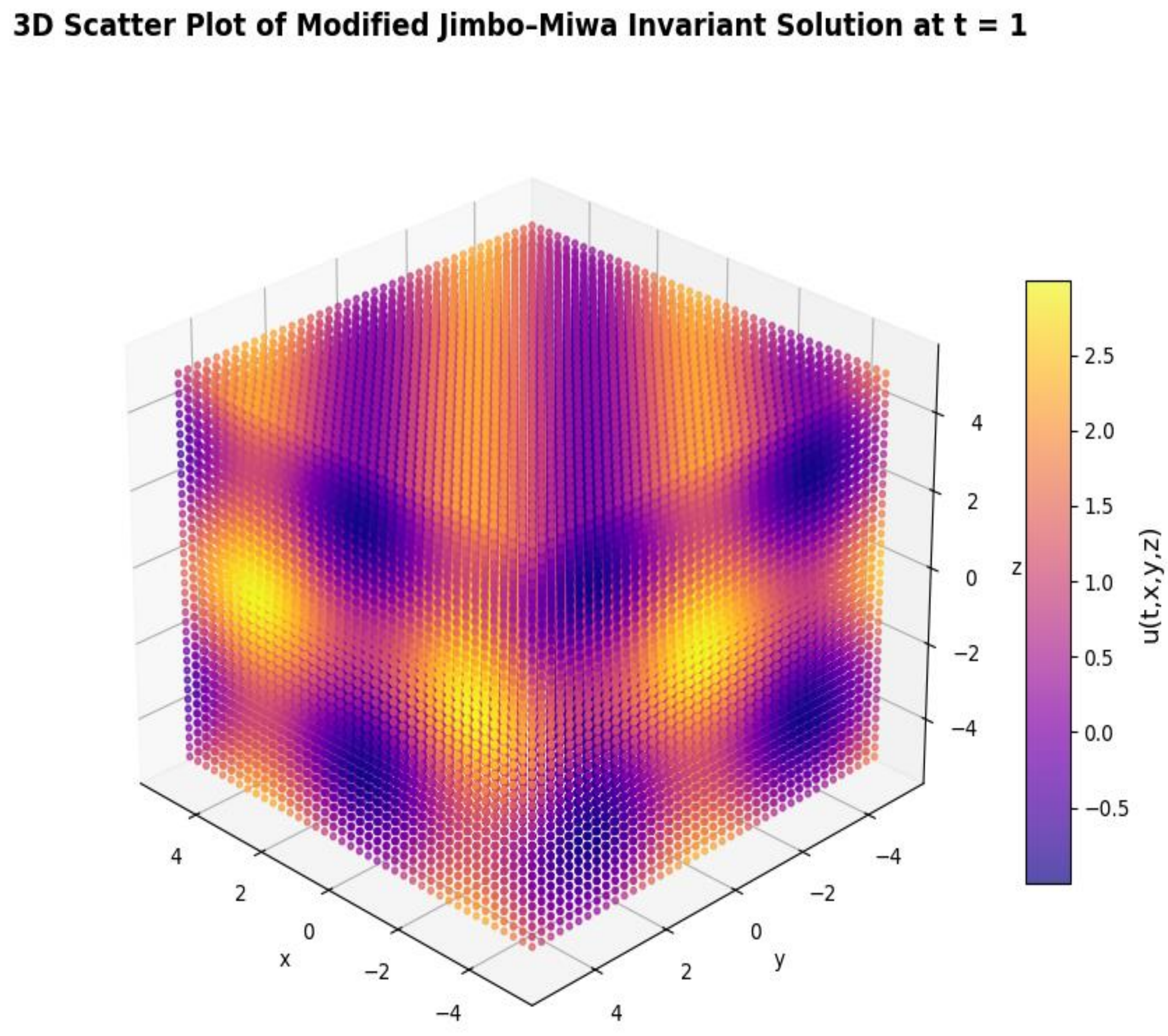
3D Scatter Plot of Modified Jimbo-Miwa Invariant Solution at t = 1

Figure 8: 3D Scatter Plot of Modified Jimbo-Miwa Invariant Solution at t = 1

The scatter plot given in Figure 8 represents the solution in full three-dimensional space. The systematic distribution of points means consistency and stability of the solution.

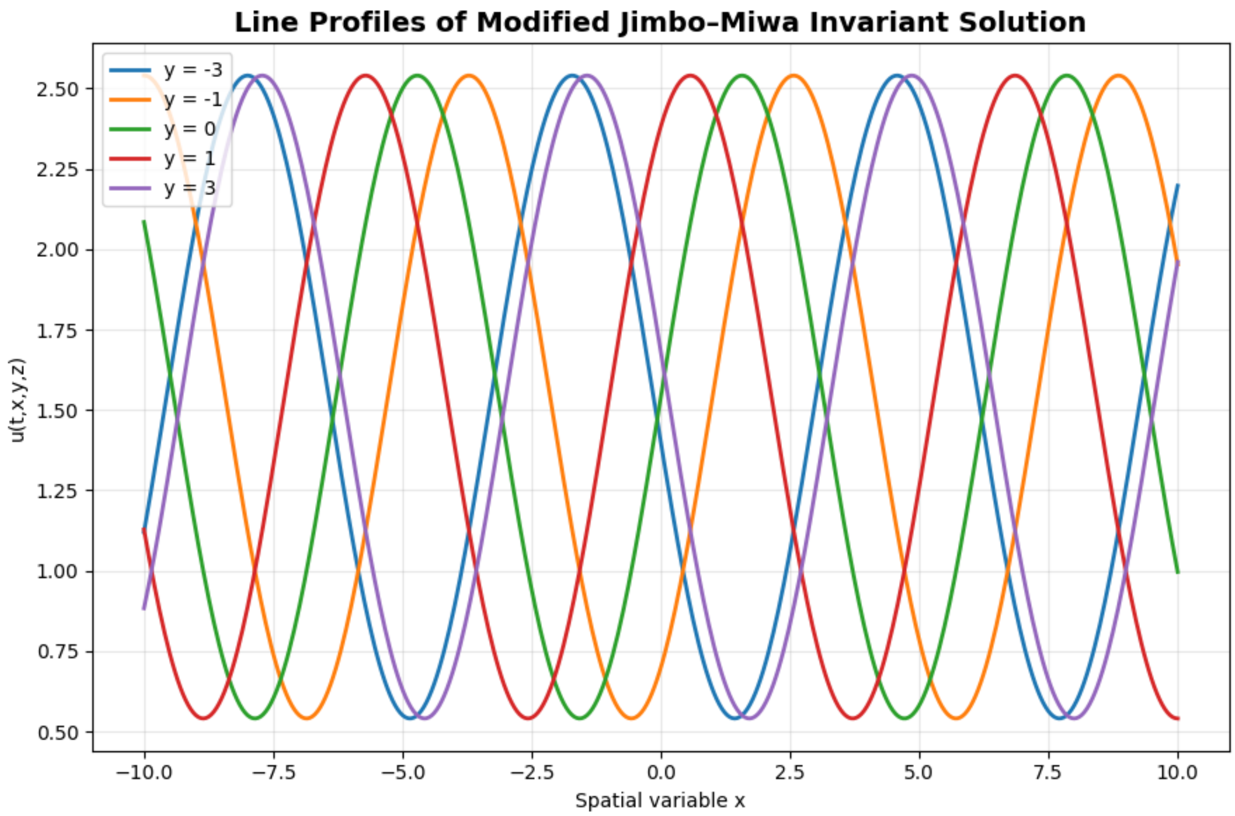

Figure 9: Line Profiles of Modified Jimbo-Miwa Invariant Solution

The line profiles show in Figure 9 oscillatory behaviour with phase shifts for different values of y. This affirms the dependence on space and periodicity of the solution.

3.4 Comparative Numerical Summary

Equation	Representative Solution	Behavior	Dimensionality
KdV	$t + \operatorname{sech}^2(x - t)$	Travelling wave	1D
JM	$t + \sin(x + t)\cos(y)$	Periodic surface	2D
Jimbo–Miwa	$t + \sin(x + y) + \cos(z)$	Multidimensional wave	3D

The comparative analysis reveals that the KdV equation generates localised travelling waves whereas the JM equation generates periodic structures in two spatial dimensions. The altered Jimbo-Miwa equation exhibits a more complex behaviour, with the extra spatial variable, which leads to multidimensional propagations of waves. These graphical results confirm the existence of the non-classical method of symmetry, which is useful in finding physically relevant invariant solutions to nonlinear PDEs.

4. Conclusion

The non-classical Lie symmetry analysis is successfully applied to the KdV equation, JM equation and modified Jimbo Miwa equation to find the invariant solutions and reduced forms of these nonlinear partial differential equations. The introduction of invariant surface conditions made it possible to construct non-classical symmetries, which are not restricted to the solutions that are available using classical Lie symmetry methods.

The analysis findings indicate the non-classical method is suitable in simplifying the complex nonlinear PDEs and producing significant exact solutions. These findings are further supported by the graphical analysis of these solutions by demonstrating the physical behaviour of the solutions obtained. The travelling wave profiles of KdV equation have demonstrated localized structures in confirming the past reputed dispersive properties. The JM equation produces periodic wave surfaces, with apparent spatial coupling, implying that there are multidimensional interactions of waves. The modified Jimbo-Miwa equation with an introduction of an additional spatial variable shows more complex behaviour with an influence on the propagation of waves and variation of phases.

Overall, the paper confirms that the non-classical symmetry analysis is a potent and versatile tool to analyse the nonlinear PDEs especially in higher dimensions. It also provides a more in-depth insight into the structure and dynamics of such equations and can reveal the usefulness of the combination of analytical and graphical methods. Future studies may be oriented towards the extension of this approach to other nonlinear models, numerical modelling, and applications to physical systems in which the multidimensional wave phenomena play an important role.

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