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RIGOROUS ANALYSIS OF A HIGHLY EFFICIENT THIRD-ORDER INTEGRATOR FOR INITIAL VALUE PROBLEMS

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Abstract

In this paper, we have theoretically studied well-organized third order numerical technique of IVP of ODE including partial derivative that has improved its competency in light of truncation error. Theoretically, integrator method of local truncation error and convergence is studied, to determine how precise and trustworthy proposed method is. Expansion of the series of Taylor is carried out that offers suitable pattern to expand and analyze function evaluation. Local truncation error, aiding with the series developed by Taylor, is an explicitly stated error to explain the order of accuracy. Linear standard test is taken in calculating the stability. Knowing behavior of method is explored under the stability. MATLAB2023a software is used to draw stability region. Stability region is also visualized to verify that the numerical method has a solution that is limited when used repeatedly and very frequently. Consistency is demonstrated that informs us that error decreases to zero as much as we shrink step size which ensures desired outcome. Convergence criteria are theoretically discussed here by proving consistency and stability. Theoretical results proving that the numerical method is stable, achieves high accuracy and provides a reliable performance. Hence, method is effective and can be used in the general class of IVP that occur in the field of ODE.

Keywords:

Local Truncation Error, Stability, Consistency, Convergence

1. INTRODUCTION

ODEs play a very important role in the development of modeling that occurs in science, engineering, physics, biology and economics. Initial value problems have analytical solutions that are either intractable or in closed form to many real-world problems. This has led to the use of numerical methods as an inseparable part of the approximation of the solutions of such equations. To formulate trustworthy numerical schemes, it is not only necessary to have computational efficiency but also a thorough grasp of their theoretical characteristics such as accuracy and stability. Error analysis is a basic element in the investigation of numerical methods. Analysis of error gives information about the impact of discretization on the accuracy of the solution and enables comparing various numerical schemes. It is traditional, in this regard, to differentiate local and global errors. The local truncation error (LTE) is the error that is introduced by the numerical method on a single step, where the previous steps are considered to be accurate. It can be used as a major indicator of ranking the accuracy. In particular, given that the LTE is expected to behave as O(h^(p+1)) then the derivation and analysis of the LTE is critical to the construction of high-accuracy numerical methods. Intimately related to analysis of errors is the idea of consistency, which ensures that the consequence approximate method is a faithful representation of the underlying differential equation as the step size approaches zero. Consistency ensures that the discrete scheme is an accurate approximation of the continuous problem, and is a component of a convergence necessity. But consistency is not enough to guarantee reliable numerical results. The other crucial property is that of stability, which addresses the action of the numerically errors during the iterative process. Stability guarantees that errors, due to truncation, round off, or initial condition perturbations, will not grow exponentially with the advancement of the computation. This is especially important in the case of stiff problems or long-term integration. One of the widely used methods to investigate stability is founded on the use of a standard linear test equation. By doing so, the so-called stability function, which defines the propagation of errors between steps, is obtained. Such an analysis leads to the notion of the stability region. It is the collection of values in the complex plane of which the numerical solution is bounded. The geometry and size of the stability region give valuable information regarding the size of the acceptable step sizes and the soundness of the method. Techniques that have wider stability regions tend to be desirable, particularly tight or highly oscillatory problems, because they can allow larger steps to be taken without loss of stability. The final aim of method is that convergence should be achieved, that is, the numerical solution should approach the exact solution when the step size tends to zero. Convergence is a crucial condition to the validity of a numerical scheme. In numerical analysis, a long-standing rule is that consistency with stability (under appropriate conditions) entails convergence. The significance of proving both properties in the analysis of a numerical method is brought out by this relationship. Based on these reflections, the formulation of useful and robust numerical schemes stipulates a holistic approach that entails the integration of error analysis, consistency, stability, and convergence. In this paper, we concentrate on the systematic exploration of these properties to a method aimed at solving ODE’s. The derivation of the LTE to obtain the order of accuracy, consistency verification, detailed stability analysis by building the stability function, and characterization of the stability region respectively, are included in the analysis. Lastly convergence is achieved through a combination of consistency and stability.

2. MATERIALS AND METHODS

Many numerical methods have designed to attain the estimated consequences of IVP's for differential equation in nature ordinary. We can consider general function form:

$$\frac{dy}{dx} = f(x_n, y_n), \quad y(x_0) = y_0 \quad ; x \in [a,b] \tag{1}$$

Interval we have given as $[a,b]$ is parted into a m uniform parts, $x_k = a + kh, (k = 0,1,2,3, \dots m)$, the step size is $h = x_{k+1} - x_k$. Solve the function $y(x)$ in a discrete series equidistant node $x_{m-1} < x_m$ to attend approximate values $y_{m-1} < y_m$.
Generally, function of two space variable also familiar as a derivative of the dependent variable regarding to independent has been calculated through integrating (1)

$$y_{n+1} = y_n + \int_{x_0}^{x_0+h} f(x_n, y_n)\, dx$$

Many equations named differential don't reach to the solution such as particular and analytical. To deal with such difficulty new innovations take place in numerical analysis. Best advantage of numerical scheme is that these have better performance than analytical. Researchers' day by day are trying to innovate many methods to obtain better consequences. But still huge number of works is needed to understand proper behaves of method. Researchers and scholars are out on the field of innovation to hone their skills. Many of them have also done a great job to construct and modified new methods. **Kandhro [2]** has Integrator Method which is

$$\begin{aligned}
w_1 &= f(x_n, y_n) \\
w_2 &= f\left(x_n + \frac{2h}{3}, y_n + hw_1\left(\frac{2}{3} - 3hf_y\right)\right) \\
w_3 &= f\left(x_n + \frac{2h}{3}, y_n - \frac{5}{6}hw_1 + \frac{3}{2}hw_2 + \frac{21}{2}h^2 w_1 f_y\right) \\
y_{n+1} &= y_n + \frac{h}{12}(3w_1 + 7w_2 + 2w_3)
\end{aligned} \tag{2}$$

This is newly Iterative Integrator Explicit Method Present in [2] having three function evaluations per time step.

3. ERROR ANALYSIS

It is main purpose to solve ordinary differential equation numerically to attain results which are as close as possible to the exact solution. Two sources of error which affect the accuracy of numerical method named as round-off (RO) and truncation (TR). Round-off error takes the place when computers can only stock numbers with limited precision and TR arises because mathematical procedures are approximated (e.g., by deserting higher-order terms). An accuracy can totally rely on blunder what size of step size is taken. We consider Taylor's series for two variables up to the fourth power of step size h, we have

$$\begin{aligned} G(x_n, y_n) = y(x) + hf + h^2\left[\frac{1}{2}f_x + \frac{1}{2}f_y f\right] + h^3\left[\frac{1}{6}f_{xx} + \frac{1}{3}\left(f_{xy} + \frac{1}{2}f_{yy}f + \frac{1}{2}f_y^2\right)f + \frac{1}{6}f_x f_y\right] \\ +h^4\left[\begin{array}{l} \frac{1}{24}f_{xxx} + \frac{1}{24}\left(3f_{xxy} + 3f_{xyy}f + 5f_y f_{xy}\right)f + \frac{1}{8}f_x f_{xy} + \frac{1}{24}f_{yyy}f^3 + \\ \frac{1}{24}\left(3f_y f^2 f_{yy} + 3f_x f f_{yy} + f_y^3\right)f + \frac{1}{24}\left(f_x f_y^2 + f_y f_{xx}\right) \end{array}\right] + O\left(h^5\right) \end{aligned} \tag{3}$$

The local truncation error is an error which spawned in a unique step of the proposed improved scheme that is documented as L_{m+1} were

$$L_{m+1} = G(x+h) - y_{m+1} \tag{4}$$

Where $G(x+h)$ is the solution obtained by Taylors's Series and y_{m+1} is used as an approximate solution. Taylor series is utilized to expand these around x and similar terms collect in h. Now we expand proposed Integrator explicit Method present in eqn (2) up to h^4, we get

$$\begin{aligned} y_{n+1} = y(x) + hf + h^2\left[\frac{1}{2}f_x + \frac{1}{2}f_y f\right] + h^3\left[\frac{1}{6}f_{xx} + \frac{1}{3}\left(f_{xy} + \frac{1}{2}f_{yy}f + \frac{1}{2}f_y^2\right)f + \frac{1}{6}f_x f_y\right] \\ +h^4\left[\begin{array}{l} \frac{1}{27}f_{xxx} + \frac{2}{9}f_{xy}ff_y + \frac{4}{3}f_y f^2 f_{yy} + \frac{1}{9}f_x ff_{yy} + \frac{1}{27}f_{yyy}f^3 \\ +\frac{7}{81}f_{xyy}f^2 + \frac{2}{81}f_{xxy}f - \frac{3}{4}f_y^3 f + \frac{1}{81}f_y f_{xx} + \frac{1}{9}f_x f_{xy} \end{array}\right] + O\left(h^5\right) \end{aligned} \tag{5}$$

Subtract (5) from (3). The proposed scheme has a local truncation error that is:

$$L_{n+1} = \frac{1}{216}\left(\begin{array}{l} f_{yyy}f^3 - 252f_y f_{yy}f^2 + \frac{65}{3}f_{xyy}f^2 + \\ \left(171f_y^3 + 3f_x f_{yy} - 3f_y f_{xy} - \frac{56}{3}f_{xyy} + \frac{65}{3}f_{xxy}\right)f \\ +9f_x f_y^2 + 3f_x f_{xy} - 3f_y f_{xx} + f_{xxx} \end{array}\right) h^4 + O\left(h^5\right) \tag{6}$$

The local truncation error (LTE) often calls with name "discretization error per time step", which helps to evaluate the order of accuracy. From LTE it is observed, LTE has 4rth order then the proposed Integrator Method in [2] has one less than the LTE in order. Therefore, the proposed Integrator Method has third order of accuracy.

4. CONSISTENCY ANALYSIS

Definition 4.1 NM together IVP with an increment function $\eta(x_n, y_n; h)$ is accepted to be consistent, if $\lim_{h\to 0}\eta(x_n, y_n; h) = f(x_n, y_n)$

Consistency of the numerical methods tells local truncation error (LTE**)** tends to zero as the step size decreases or $h \to 0$**.** Therefore, newly proposed scheme has increment function as

$$\eta(x_n, y_n; h) = \frac{1}{12}(3w_1 + 7w_2 + 2w_3)$$

Proceeds $\lim_{h\to 0}$ on both side

$$\begin{aligned}\lim_{h\to 0}\eta(x_n, y_n; h) &= \frac{1}{12}\lim_{h\to 0}(3w_1 + 7w_2 + 2w_3)\\ &= \frac{1}{12}\lim_{h\to 0}\left(\begin{array}{l} 3f(x_n, y_n) + 4f\left(x_n + \frac{2h}{3}, y_n + hw_1\left(\frac{2}{3} - 3hf_y\right)\right) + \\ 5f\left(x_n + \frac{2h}{3}, y_n - \frac{5}{6}hw_1 + \frac{3}{2}hw_2 + \frac{21}{2}h^2 w_1 f_y\right)\end{array}\right)\\ &= f(x_n, y_n)\end{aligned}$$

Hence, the Proposed Integrator Method [2] is proven to be consistent with at least **third order accuracy.**

5. LINEAR STABILITY ANALYSIS

Dahlquist's test problem is to be considered for verifying the stability of the Method in [2]

$$\frac{dy}{dx} = q\,y(x); \quad y(0) = y_0, \quad q \in C$$

Whereas q is a complex constant i.e, $q \in C$.

After executing Method (2) on this test, we attain

$$w_1 = q\,y_n$$

$$w_2 = q_n\left[y_n + \frac{2}{3}hw_1 - 3h^2 w_1 f_y\right];$$

$$\Rightarrow w_2 = q_n y_n\left[1 + \frac{2}{3}(hq) - 3(hq)^2\right]; \qquad \therefore f_y = q$$

$$w_3 = q_n\left[y_n - \frac{5}{6}hw_1 + \frac{3}{2}hw_2 + \frac{21}{2}h^2 w_1 f_y\right]$$

$$\Rightarrow w_3 = q\,y_n\left[1 + \frac{2}{3}(hq) + \frac{23}{2}(hq)^2 - \frac{9}{2}(hq)^3\right]$$

Substituting above all values in (2) in right side of y_{n+1}, we get

$$y_{n+1} = y_n + (hq)y_n + \frac{1}{2}(hq)^2 y_n + \frac{1}{6}(hq)^3 y_n - \frac{3}{4}(hq)^4 y_n$$

Factor y_n;

$$y_{n+1} = y_n R(z)$$

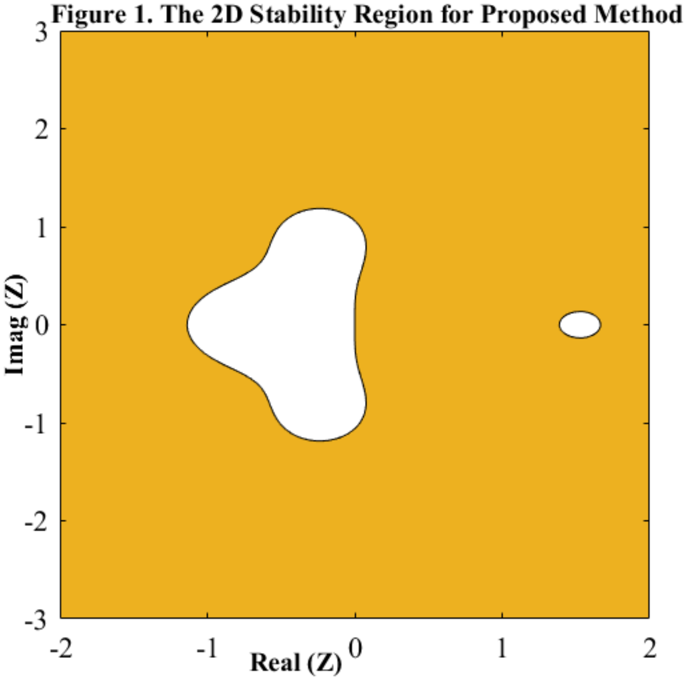
$$R(z) = 1 + (hq) + \frac{1}{2}(hq)^2 + \frac{1}{6}(hq)^3 - \frac{3}{4}(hq)^4$$

This is Stability Function $R(z)$ for iterative integrator method in [2] which reflects the third-order accuracy of the method. When errors are introduced then linear stability is checked to know how numerical method behaves. In particular, it expresses that whether those errors decay, remain bounded, or grow uncontrollably during the computation. Stability controls error propagation.

6. REGION OF ABSOLUTE STABILITY

Region of Integrator method is calculated and plotted in MATLAB. A complex grid of values $z = x + iy$ is generated, and the stability function $R(z)$ is evaluated over this domain. The set of points satisfying $|R(z)| \leq 1$ is identified and plotted to illustrate the stability region in the complex plane.

Figure 1. The 2D Stability Region for Proposed Method



This stability region is a fundamental concept used to determine whether a numerical method produces bounded solutions when applied repeatedly. Figure 1 illustrates the two-dimensional region of absolute stability for the proposed numerical method in the complex plane. The real part $Re(z)$ is demonstrated on horizontal line of axis, while the imaginary part $Im(z)$ is demonstrated on vertical axis of line, where $z = hq$. The white (unshaded) regions indicate the set of values for which the stability condition $|R(z)| \leq 1$ is satisfied. These regions represent the domain when solution approximately remains stable and keep bounded. The orange (shaded) region corresponds to values of z for which $|R(z)| \leq 1$ indicating instability, where numerical errors may grow exponentially.

7. CONVERGENCE OF THE METHOD

A method is to be numerically convergent when both consistency and stability justified. The necessary condition for convergence of a numerical method is consistency, which guarantees that the local discretization error vanishes as step-size very nearly approaches zero. However, convergence is achieved only when consistency is combined with stability. Therefore, above said words the accelerated proposed method is converges.

8. CONCLUSION

In this work, the construction and detailed mathematical analysis of a numerical method for solving initial value problems of ordinary differential equations. We theoretically examined well-organized third order numerical technique for IVP of ODE’s including partial derivative which has enhanced its competency regarding truncation error. The main focus is not here to derive scheme but here is justify its theoretically behave which is analyzed by systematic investigation of error and convergence. The LTE has derived perfectly which endorsing accuracy by identify its order. Stability is investigated for knowing behavior of method. Stability region is envisioned to check numerical method possess a bounded solution when applied frequently and repeatedly. Convergence criteria theoretically discussed here by proving consistency and stability. Theoretically findings showing that the numerical method is stable, attains high accuracy and gives reliable performance. Therefore, method is efficient and applicable for extensive class of IVP arising in the area of ODE’s.

REFERENCES

1. Ashiribo Senapon Wusu, Moses Adebowale Akanbi and Solomon Adebola Okunuga ‘’A Three-Stage Multiderivative Explicit Runge-Kutta Method’’, American Journal of Computational Mathematics, 2013, 3, 121-126
2. Muhammad Daud Kandhro**,** (2026) A Highly Efficient Third-Order Integrator for Initial Value Problems -vol. 4 No. 2 (2026): (February 2026) Annual Methodological Archive Research Review
3. Owolanke, A.O., Uwaheren, O. And Obarhua, F.O. (2017) An Eight Order Two-Step TaylorSeries Algorithm for The Numerical Solutions of Initial Value Problems of Second Order Ordinary Differential Equations. Open Access Library Journal, 4: E3486.
4. Butcher, John Charles. Numerical methods for ordinary differential equations. John Wiley & Sons, 2016
5. A Third Runge Kutta Method Based on a Linear Combination of Arithmetic Mean, Harmonic Mean and Geometric Mean
6. Mukaddes Okten Turacı1, 2·Turgut ¨Ozis¸1 ‘’Derivation of Three-Derivative Runge-Kutta Methods’’ Springer Science Business Media New York 2016
7. Wazwaz, Abdul-Majid. "A modified third order Runge-Kutta method." Applied Mathematics Letters 3.3 (1990): 123-125.
8. Abdul Majid Wazwak ‘’A Modified Third Order Runge Kutta Method” Al Quds university college of science and technology (1990).
9. Rao V. Dukkipati "Numerical Methods " First Edition 2010
10. Burden, Richard L., and J. Douglas Faires. "Numerical analysis. 2001." Brooks/Cole, USA (2001).
11. Peter V. O’Neil "Advanced Engineering Mathematics" 7th edition
12. Akanbi, M.A., (2010), On 3-stage geometric explicit Runge–Kutta method for singular autonomous initial value problems in ordinary differential equations. Springer-Verlag.
13. Butcher, J.C., (1964), On Runge-Kutta processes of high Order, J. Austral. Math. Soc., 4
14. Fatunla, S.O., (1991), Numerical Methods for IVPs in ODEs. Academic Press, New York.
15. Dr Najmuddin Ahmed, Study of Numerical Accuracy of Runge-Kutta Second, Third and Fourth Order Method

16. Hairer, E . ; Narsett, S. P. and Wanner, G. (1993): Solving Ordinary Differential Equations. I. Nonstiff problems, Vol. 8 of Springer Series in Computational Mathematics Springer – Verlag (Berlin), Second Ed.
17. Hundsdofer , W. and Verwer , J. G. (2003) : Numerical Solution of Time dependent Advection – Diffusion – Reaction Equations , Vol. 33 of Springer Series in Computational Mathematics . Springer (Berlin)
18. Kaw, Autar; Kalu , Egwu (2008) : Numerical Methods with Applications (1st Ed.), www.autarkaw.com.
19. Runge – Kutta – Fehlberg Type Procedure on Two Nodes for Numerical Integration of Systems Differential Equations . Dumitras , Daria Elena Automat . Comput. Appl. Math, Vol. 2, pp 139 – 143, Math. Sci. Net.
20. Test results on Initial Value Methods for Non – Stiff Ordinary Differential Equations W. H. Enright; T. E. Hull SIAM Journal on Numerical Analysis, Vol. 13 No. 6. (Dec. 1976), pp 944 – 961, Jstor.
21. Verner , J. H. (1991) : Some Runge – Kutta Formula Pairs . SIAM J. Numer . Anal. Vol. 28 , No. 2 , pp 496 – 511, Math . Sci. Net.