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DEVELOPMENT AND ANALYSIS OF A HIGH-ACCURACY THIRD-ORDER RUNGE–KUTTA METHOD FOR NON-AUTONOMOUS INITIAL VALUE PROBLEMS

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Article Info

Abstract

New class of explicit and single step third order improved scheme developed to numerically integrate initial value problems for differential equations in nature ordinary. The stability requirements of the enhanced plan are explored, and stability region of the same is plotted. Third order accuracy of the improved scheme is also confirmed by error analysis performed. The partial derivative with respect to the space dependent variable has been introduced inside the function evaluation which boasts the convergence and efficiency of the proposed scheme. Few problems have been tested to determine absolute errors named as last and maximum. Approximate solutions are provided and demonstrate the accuracy the better scheme. The performance of the improved and the existing schemes with same order of local accuracy are discussed. Based on the results, the improved scheme gives the better result in comparison of other available schemes in literature. With the help of software MATLAB 2023a, the numerical results and graphical representation of the newly proposed scheme are justified.

Keywords: *Numerical Method, Third-Order, Local Truncation Error, Consistency, Stability.*

1. INTRODUCTION

Many of the issues which occur in science, mathematical physics, and engineering are naturally expressed as differential equations. They are equations which are the basic tools to explain the behavior of dynamic systems where one or more quantities change as a result of time or space. Population dynamics, radioactive decay processes, mechanical vibrations, electrical circuit analysis, fluid flow behavior and chemical reaction kinetics are typical examples. In most of such instances, the relationships of control are formulated in the form of ordinary differential equations (ODEs), especially where the system is a part of one independent variable. Differential equations have been used to offer a common framework to model such a wide range of physical phenomena, as stressed in standard texts by Dennis G. Zill, Warren S. Wright and Peter V. O'Neil. Ordinary differential equations are often expressed in the form of initial value problems (IVPs), in which the state of the system is known at some starting point, and it is desired to find out how the system will behave in the future. These equations could be either autonomous with the independent variable not being explicitly present in the formulation or non-autonomous with the independent variable explicitly present in the formulation. Such equations play a crucial role in solving real world phenomena and predicting them accurately and efficiently. Even though analytical methods can give precise answers to some types of differential equations, they can only give answers to relatively simple or idealized problems. Numerous real-world issues contain nonlinearities, initial condition, or variable coefficients and thus closed-form solutions are hard or impossible to arrive at. As a result, the use of numerical procedures has become inevitable when trying to approximate solutions of the ordinary differential equations. John Charles Butcher (2016) and Erwin Kreyszig (2011) discuss both analytical and numerical methods of solving such equations, emphasizing the role of numerical schemes when the exact solution is not possible. Methods of solving IVPs using numerical methods can be broadly divided into single-step and multi-step methods. Single-step methods calculate the solution at one of the new points using only the information of the current step which makes them easy to implement and can be used in many cases. The simplest one is the method of Euler, which gives a simple approximation, but can be characterized by low accuracy and stability. To overcome these shortcomings, better and refined versions of the method by Euler have been created, with more function evaluations to improve performance. Single step methods are more advanced such as explicit and implicit Runge-Kutta methods, which are highly utilized because of their accurate and stable characteristics. Of these, Runge-Kutta methods are one of the most important developments in numerical analysis. These approaches have higher-order accuracy without involving higher-order derivatives and are especially appealing to practice. The overall concept of RungeKutta methods is to solve the equations of motion by integrating a series of judgments of the differential equation in one step. The method can be used to achieve better accuracy than the simpler ones whilst maintaining ease of implementation. Specifically, third-order Runge-Kutta methods have received a lot of interest because they offer an acceptable compromise between computational efficiency and accuracy. These schemes are more precise than the second-order schemes and avoid the higher complexity of higher-order schemes like fourth-order Runge-Kutta. Consequently, third-order methods are particularly applicable to any application where moderate accuracy is needed and computational resources are scarce. Different forms of third-order Runge-Kutta methods have been suggested over the years with each attempting to maximise a given property, including stability, error control and the cost of computation. Multi-step methods on the other hand use multiple prior points to give an approximation of the solution of the next point. They encompass Adams Bashforth and Adams

Moulton methods, which rely on interpolation formulas, and predictor corrector methods, which build up approximate solutions. Rao V. Dukkipati (2010) explains that these methods have been reported to be effective in reducing the computation time without sacrificing accuracy. However, they are often dependent on initial values that are obtained by using one-step methods and could be more complex to implement. As a consequence, numerical methods are quite popular in solving differential equations, particularly when analytical solutions are not known. Initial value problems are particularly relevant in this context and a lot of science and engineering applications are based on them. There are many numerical methods to determine the exact approximate solution to ordinary differential equations. These approaches are not new and can be found in the literature, such as Kreyszig et al. (2011), Dukkipati et al. (2010), Owolanke et al. (2017), Mukaddes Okten et al. and many others who have worked on the development of numerical solution methods. There are many numerical methods, but research on developing improved techniques is ongoing. The objective is to find schemes that are more accurate, stable and faster to compute. In this context, numerical methods are thought to be more practical and efficient than the analytical methods, particularly in terms of convergence. This has attracted countless efforts to improve existing techniques and develop new techniques that will be able to solve complex differential equations. The other factor that is significant in numerical analysis is convergence which ensures the approximate solution is close to the exact solution as the step size tends to zero. Consistency and stability are other important properties that impact the accuracy of a numerical method. A third-order Runge Kutta scheme should have these attributes and be efficient. Therefore, the evaluation and analysis of other numerical methods, with error estimates and computational cost, are crucial to this field.

In practice, it is usual to compare numerical solutions in terms of absolute error, maximum error over an interval and time of computation. The metrics provide an indication of the efficiency of a method and helps in the selection of an appropriate method for a given problem. In many cases, there is no analytical solution to complex systems; hence such comparison studies are crucial for testing newly developed numerical methods. In summary, ordinary differential equations are an important part of pure and applied mathematics due to the variety of applications in modeling problems of the real world. While analytical approaches are limited, numerical approaches provide an effective and versatile means for obtaining approximate solutions. Among these, third-order Runge-Kutta methods play a prominent role, in that they offer a reasonable compromise between accuracy and efficiency. Current research is further broadening these techniques, which are contributing to the advancement of numerical analysis and its many applications in science and engineering.

2. DERIVATION OF THE IMPROVED SCHEME

We have lots of numerical schemes to obtain solution of numerous initial value problems especially for ordinary differential equation, so for this we consider general form of ordered first ordinary differential equation:

$$y'(x) = f(x_m\,, y_m)\,; \quad y(x_0) = \rho \tag{1}$$

We assumed closed interval called as integration interval of $x \in [x_0, x_m]$ to rectify existence of uniqueness of the solution of (1). Whereas exact or true solution is represented by $y(x_n)$ and y_n is used for numerical solution, h is recognised as step size, which is $h = \frac{1}{M}(x_m - x_0)$ where $N = 1,2,3,\ldots$

Generally, real space of vectors in n-dimensional is acknowledged as y, y_0, y_n and f, which has been sought by integrating (1) from x_0 to $x_0 + h$ in the form

$$y_{m+1} - y_m = \int_{x_0}^{x_0+h} f(x_m, y_m)dx$$

or, in equivalent form

$$y_{m+1} - y_m = h\sum_{m=1}^{p} q_m w_m$$

Where q_m is utilized for parameters and w_m is utilized for slopes

We take different m, we are taking m as 3, i.e, $m = 3$, then it can be molded as

$$y_{m+1} - y_m = h\,[q_1 w_2 + q_2 w_2 + q_2 w_3] \tag{2}$$

$$w_1 = f(x_m, y_m)$$

$$w_2 = f\big(x_m + \alpha_2 h, y_n + h(\beta_{21} w_1) + h^2\,(\gamma_{21} w_1) f_y\big)$$

$$w_3 = f\big(x_m + \alpha_3 h, y_n + h(\beta_{31} w_1 + \beta_{32} w_2) + h^2\,(\gamma_{31} w_1) f_y\big)$$

Most commonly Taylor series is utilized to acquire the value of slope w_2 and w_3. The Taylor series expansion of $Y(x_m, y_m)$ is;

$$Y(x_m, y_m) = y(x) + hf + h^2\left[\frac{1}{2}f_x + \frac{1}{2}f_y f\right] + h^3\begin{bmatrix}\frac{1}{6}f_{xx} + \frac{1}{6}(2f_{xy} + f_{yy}f)f \\ +\frac{1}{6}(f_y^2 f + f_x f_y)\end{bmatrix}$$

$$+\frac{h^4}{24}\begin{bmatrix}f_{xxx} + 3(f_{xxy} + f_{xyy}f)f + (5f_y f + 3f_x)f_{xy} + f_{yyy}f^3 \\ +4f_y f^2 f_{yy} + 3f_x f f_{yy} + f_y^3 f + f_x f_y^2 + f_y f_{xx}\end{bmatrix} + O(h^5) \tag{3}$$

Eqn. (3) is Taylor's series which is narrated as an infinite sum of terms which can be expressed in terms of the function's derivatives at a single point.

3. THE ORDER-CONDITIONS DERIVATION OF THE METHOD

Here, we utilize the Taylor's series approach for evaluating the parameters of (2). Very first we have to determined OCs of the numerical integrator present in (2). Whereas w_2 and w_3 are expended by given Taylor's series. Secondly, we insert the result of w_1, w_2 and w_3 into (2), then power of h comparing upto h^3 with that of (3) to acquire the order conditions which are in the form of system of nonlinear equations:

$$q_1 + q_2 + q_3 = 1 \qquad \alpha_2 q_3 \beta_{32} = \frac{1}{6} \qquad \alpha_2 q_2 + \alpha_3 q_3 = \frac{1}{2}$$

$$\frac{1}{2}(\alpha_2^2 q_2 + \alpha_3^2 q_3) = \frac{1}{6} \qquad q_2\beta_{21} + q_3\beta_{31} + q_3\beta_{32} = \frac{1}{2} \tag{4}$$

$\alpha_2 q_2 \beta_{21} + \alpha_3 q_3 \beta_{31} + \alpha_3 q_3 \beta_{32} = \frac{1}{3}$ $\qquad$ $q_2 \gamma_{21} + q_3 \gamma_{31} + q_3 \beta_{21} \beta_{32} = \frac{1}{6}$

$\frac{1}{2}(q_2 \beta_{21}^2 + q_3 \beta_{31}^2 + q_3 \beta_{32}^2) + q_3 \beta_{31} \beta_{32} = \frac{1}{6}$

This Eqn (4) represents a nonlinear system of equations having 8 equations with 10 unknowns' quantities. Furthermore, we will have to search out, system has which type of solution. To solve such confusion, parameters are to be founded. One of the solutions of the solution is founded as;

$$
\begin{aligned}
w_1 &= f(x_m, y_m) \\
w_2 &= f\left(x_m + \frac{2}{3}h, y_m + \frac{2}{3}hw_1 - h^2 w_1 f_y\right) \\
w_3 &= f(x_m + \frac{2}{3}h, y_m + h(-\frac{2}{3}w_1 + \frac{4}{3}k_2) + 3h^2 w_1 f_y) \\
y_{m+1} &= y_m + \frac{h}{16}(4w_1 + 9w_2 + 3w_3)
\end{aligned}
\tag{5}
$$

After acquiring new robust RK-family type improved scheme (5). To check more reliability of new proposed scheme, we further investigate scheme for accuracy as well as stability.

4. ERROR ANALYSIS

The local truncation error is an error which spawned in a unique step of the proposed improved scheme that is documented as L_{m+1} were

$$L_{m+1} = y(x+h) - y_{m+1} \tag{6}$$

Where $y(x)$ is the theoretical solution and y_{m+1} is used as an approximate solution. Taylor series is utilized to expand these around x and similar terms collect in h. The proposed scheme has a local truncation error that is:

$$L_{m+1} = \frac{1}{216}\begin{bmatrix} f_{yyy}f^3 + (-81 f_y f_{yy} + 21 f_{xyy})f^2 + \\ 3(21 f_y^3 + f_x f_{yy} - f_y f_{xy} + 7 f_{xxy} - 6 f_{xyy})f \\ +3\{(3f_y^2 + f_{xy})f_x - f_y f_{xx}\} + f_{xxx} \end{bmatrix} h^4 + O(h^5) \tag{7}$$

5. CONSISTENCY ANALYSIS

Definition 5.1 Given an initial value problem $y'(x) = f(x_m, y_m)$; $y(x_0) = \rho$; an improved scheme with an increment function $\vartheta(x_m, y_m; h)$ is called to be consistent, when

$$\lim_{h \to 0} \vartheta(x_m, y_m; h) = f(x_m, y_m) \tag{8}$$

Therefore, newly proposed scheme has increment function as

$$\vartheta(x_m, y_m; h) = \frac{1}{16}(4w_1 + 9w_2 + 3w_3) \tag{9}$$

By utilizing $\lim_{h\to 0}$ both sides

$$\lim_{h\to 0}\vartheta(x_m, y_m; h) = \lim_{h\to 0}\frac{1}{16}(4w_1 + 9w_2 + 3w_3)$$

$$= \lim_{h\to 0}\frac{1}{16}\left(4f(x_m, y_m) + 9\left[f\left(x_m + \frac{2}{3}h\,, y_m + \frac{2}{3}hw_1 - h^2 w_1 f_y\right)\right]\right.$$
$$\left. + 3\left[f\left(x_m + \frac{2}{3}h\,, y_m + h\left(-\frac{2}{3}w_1 + \frac{4}{3}w_2\right) + 3h^2 w_1 f_y\right)\right]\right)$$

$$\lim_{h\to 0}\vartheta(x_m, y_m; h) = f(x_m, y_m) \tag{10}$$

Hence, the newly proposed scheme with at least **third order accuracy** is proven to be **consistent.**

6. LINEAR STABILITY ANALYSIS

Dahlquist's test problem is to be considered for verifying the stability of the scheme
$\frac{dy}{dx} = \lambda\, y(x);$ whereas $y(0) = \eta\,,\ \lambda \in C$

After executing scheme (5) on this test, we attain stability function with linear stability region which is displayed by unfilled region in fig 1.

$$w_1 = \lambda y_n;$$
$$w_2 = \lambda y_m\left[1 + \frac{2}{3}h\lambda - (h\lambda)^2\right];$$
$$w_3 = \lambda y_m\left[1 + \frac{2}{3}h\lambda + \frac{35}{9}(h\lambda)^2 - \frac{4}{3}(h\lambda)^3\right];$$

Substitute all values in (5), polynomial form of stability function is derived
$$R(Z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{4}z^4 \text{ where } z = h\lambda$$

Figure 1. The 2D Stability Region for Proposed Method

7. NUMERICAL EXPERIMENTS

This section examines several IVP's in ordinary differential equations to illustrate performance of the newly proposed scheme in comparison with other methods of the same order. Absolute error (maximum and last) within CPU time is computed for evaluating effectiveness of scheme. The results achieved and compared by utilizing proposed method and two standard methods such as the Runge–Kutta method based on the harmonic mean (RK3HM) and third-order Heun method (H3M), as defined below.

Table 1: Shows Errors and CPU Values

Problem 1. Nonlinear Cauchy problem

$$\frac{dy}{dx} = xy^3 - y, \quad y(0) = 1, \qquad Exact = \frac{2}{\sqrt{2+4x+2e^{2x}}}$$

Step-Size /Methods		RK3HM	HEUN	Proposed
0.1	Maximum-Error	2.2406e-04	4.3314e-05	2.3020e-05
	Last-Error	2.1363e-04	4.0424e-05	1.5667e-05
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0
0.05	Maximum-Error	q5.1698e-05	5.2797e-06	4.3411e-06
	Last-Error	4.9489e-05	4.9295e-06	3.8685e-06
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0
0.025	Maximum-Error	1.2497e-05	6.4999e-07	6.1161e-07
	Last Error	1.1988e-05	6.0711e-07	5.6819e-07
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0
0.0125	Maximum-Error	3.0773e-06	8.0582e-08	8.0061e-08
	Last-Error	2.9553e-06	7.5283e-08	7.5481e-08
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0

Figure 2. Absolute Error with h=0.1

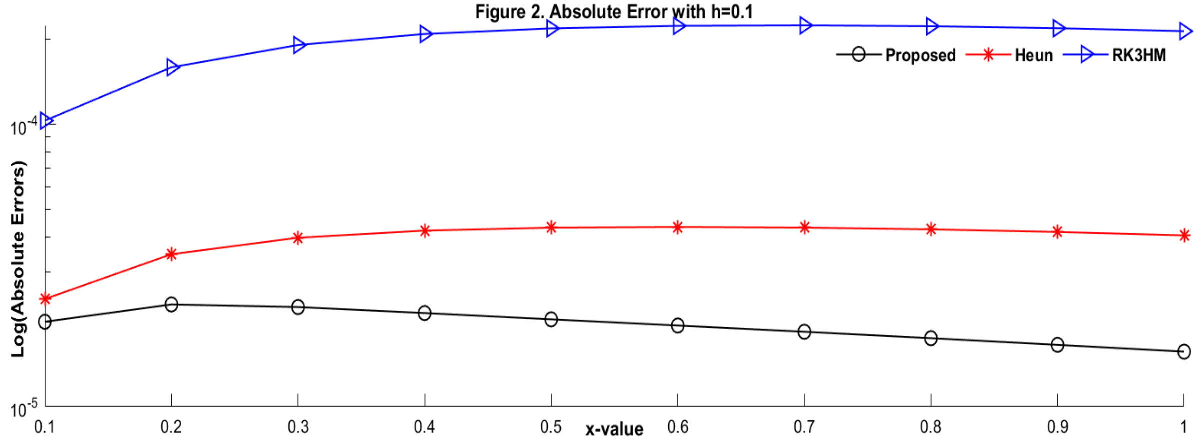
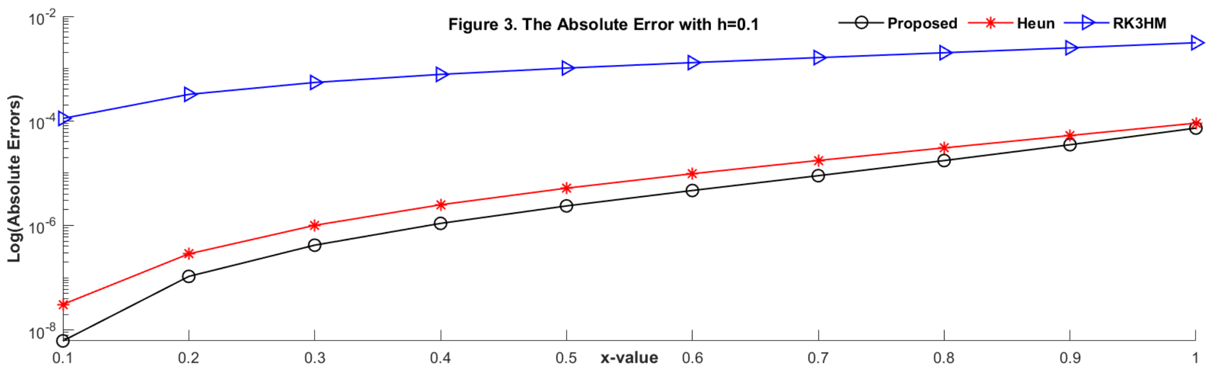


Table 2: Shows Errors and CPU Values

Problem 2. Nonlinear Cauchy problem $\frac{dy}{dx}=x^2y,\quad y(0)=1,\quad Exact=e^{\frac{x^3}{3}}$				
Step-Size/Method		RK3HM	HEUN	Proposed
0.1	Maximum-Error	0.0031	9.0292e-05	7.2595e-05
	Last-Error	0.0031	9.0292e-05	7.2595e-05
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0
0.05	Maximum-Error	8.2411e-04	1.1713e-05	9.7986e-06
	Last-Error	8.2411e-04	1.1713e-05	9.7986e-06
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0
0.025	Maximum-Error	2.1194e-04	1.4897e-06	1.2692e-06
	Last Error	2.1194e-04	1.4897e-06	1.2692e-06
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0
0.0125	Maximum-Error	5.3744e-05	1.8777e-07	1.6137e-07
	Last-Error	5.3744e-05	1.8777e-07	1.6137e-07
	Time (CPU)	0.000e+0	0.000e+0	0.000e+0



7. RESULTS AND DISCUSSIONS

Third-order improved scheme is a newly developed scheme that is an effective and reliable scheme to solve Cauchy problems in computational and applied mathematics. Numerical experiments were conducted to evaluate its performance with various step sizes, that is, 0.1, 0.05, 0.25 and 0.125. In each case, the key performance indicators that included absolute maximum error, error at the last nodal point, and CPU execution time (in seconds) were measured and tabulated. Such measures provide an all-inclusive foundation to assess the correctness and computing efficiency of the proposed method. A closer look at the tabulated results will show that the proposed scheme, in comparison with other numerical methods of the same order of precision, will always have much smaller absolute maximum errors and final point errors. Notably, the accuracy is improved without raising the computation cost since the CPU time taken is similar on average. In addition, the numerical solutions that are obtained with the improved method are significantly closer to the actual (analytical) solutions. The proposed scheme proves to be more precise and reliable than the known algorithms (Runge-Kutta method with harmonic mean of three quantities (RK3HM) and the third-order Heun method (H3M)) in all of the tested step sizes. Besides the benefits of accuracy, the proposed third-order scheme has a faster convergence rate. The approximation solution closes very fast in relation to the exact solution as the step size is reduced and this is a good sign of a high level of numerical stability and efficiency. A comparative study reveals that the convergence rate of the enhanced method is higher than those of RK3HM and classical third-order RungeKutta method. Therefore, the proposed method is one of the most useful tools to solve Cauchy problems in ordinary differential equations in the category of numerical schemes with the same order of local accuracy, providing an optimal trade-off between accuracy, convergence rate, and computing time.

8. CONCLUSION

A new third-order explicit method for solving Cauchy problems in ordinary differential equations is developed in this paper. The scheme is designed to be highly accurate (third-order) and simple to implement. A linear stability analysis is performed, and the stability region shows that the method is stable provided a suitable step size is chosen. The effectiveness of the method is evaluated by applying it to a number of numerical examples, comparing the results with those obtained using standard methods. The findings are presented in tables, which show maximum error, final error, and time for various step sizes. These tests demonstrate that the proposed scheme has lower absolute errors. This is also visually supported in the graphical illustrations, where it can be seen to be more accurate than several other schemes, such as the third-order Heun method and the Runge-Kutta method based on the harmonic mean, which have relatively rapid error growth. The results obtained from the considered Cauchy problems suggest that the proposed scheme is extremely efficient, especially with respect to local accuracy. It can provide accurate solutions with minimum error, making it a promising method for real-world applications. As a result, this enhanced scheme can be regarded as a reliable and efficient method for solving the Cauchy problems that are commonly found in computational and applied mathematics.

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