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A HYBRID CUBIC B-SPLINE APPROACH FOR THE NUMERICAL TREATMENT OF THE SINE-GORDON EQUATION

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Article Info

Abstract

In this article, a hybrid cubic B-spline collocation method has been proposed to find the numerical solutions of a well-known nonlinear Sine-Gordon equation. The finite difference scheme has been employed to discretize the time derivative, whereas cubic B-spline functions are used for spatial discretization. The efficiency of the applied method is checked through some test problems. Numerical outcomes are compared with the exact solutions available in the literature which illustrate the effectiveness of the proposed method. It can easily be concluded that the results obtained are reliable and consistent with those found in earlier research.

Keywords:
Hybrid cubic B-Spline, system of equations, Sine-Gordon equation, spatial variable, finite difference scheme.

1. Introduction

Nonlinear evolution equations have broad applications across many aspects of engineering, physics, chemistry, and biology which include chemical reactions, gas and fluid mechanics, elasticity, optical fibers, relativity, solid-state and plasma physics, ecology and biomechanics among others [1]. Analytical and numerical studies of traveling wave equations have always been a popular area of study among researchers. The research work in question takes into consideration one-dimensional sine-Gordon equation that is solved.

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial y^2} - \sin(v), \qquad y \in (L_1, L_2), \qquad t \geq 0, \tag{1}$$

with initial conditions

$$v(y,0) = \phi_1(y), v_t(y,0) = \phi_2(y). \tag{2}$$

The boundary conditions in Dirichlet sense are expressed as

$$v(L_1,t) = \psi_1(t), v(L_2,t) = \psi_2(t), t \geq 0. \tag{3}$$

The nonlinear sine-Gordon equation has a wide range of applications in the propagation of fluxion in Josephson junctions [1], in differential geometry, in the stability of fluid motion, in nonlinear physics, and in applied sciences. The sine-Gordon equation (1) is a specific instance of the Klein-Gordon equation [2], [3], [4], which is crucial in numerous scientific domains, including solid-state physics and nonlinear optics.

$$\frac{\partial^2 v}{\partial t^2} - \alpha \frac{\partial^2 v}{\partial y^2} + g(v) = f(y,t) \tag{4}$$

Where α is a constant and $g(v)$ is a nonlinear force and α is a constant. Various approaches in literature have been suggested in solving numerically sine-Gordon equation. Ben-Yu *et al.*[2], gave two difference schemes, and Bratsos and Twizell, the method of lines transformation of the initial/boundary value problem, which results when (1) is turned into a first-order nonlinear initial value problem. A fourth-order stable DIRKN method in combination with a compact finite difference approximation of fourth order was described by Mohebbi and Dehghan [5]. Kuang and Lu [6] have developed classes of finite difference scheme of generalized sine-Gordon equation. Whereas a in [7], Bratsos and Twizell derived a class of difference schemes in which the time and space derivatives are replaced by a finite-difference approximation of the derivatives at the points to give a linear algebraic system. Wei [8] used the discrete singular convolution algorithm to compute the integral in (1). This technique has been used by Batiha *et al.* [9] to devise a variational iteration procedure to arrive at an approximate analytical solution of the sine-Gordon equation with no discretization. An explicit solution of sine-Gordon equation on the entire real line was presented in [10]. To obtain a numerical solution to (1), in a 3 time level recurrence relation, Bratsos [11] approximated the matrix exponential term with a fourth-order rational approximation. Later on , Dehghan and Shokri [12] tackled the equation as a special collocation points solve and attacked the solution as a radial basis functions solve. Rashidinia and Mohammadi [13] tackled the equation as two implicit-finite difference schemes employing spline collocation approach. Li-Min and Zong-Min created a meshlass strategy based on multi quadric quasi-interpolation and need a polynomial to enhance the authenticity of this method. Jiang and Wang [14] suggest another meshless scheme that uses high accuracy MQ quasi interpolation without the use of polynomial. Kaya suggested a modified decomposition technique of the approximate and solution of nonlinear sine-Gordon equation with the power series method. Uddin *et al.* [15] employed the radial basis functions to approximate the numerical solution of the Sine-Gordon equation

Table 1: The numerical values of $B_j(y)$ and its derivatives.

y	y_{j-2}	y	y_j	y_{j+1}	y_{j+2}
$B_j(y)$	0	1	4	1	0
$B_j'(y)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
$B_j''(y)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

Among the key properties of B-splines is minimum compact support and smoothness that enable it to solve numerically partial differential equations (PDEs) regarding linearity and non-linearity in nature. When B-spline is applied along with collocation method, a very simplistic process of solving differential equations can be achieved. It also gives a spline function which comes in handy in getting solutions at any point of the domain that one chooses, unlike in the instance of finite-difference schemes, where one can only get the solutions at the fixed knots. The proposed technique determines numerical approximations of the sine-Gordon equation by a space B-spline Hybrid cubic B-spline technique. Sine-Gordon equation is transformed into a PDEs system. This is then further simplified into an ordinary differential equation system by using HCBCM. The nonlinear sine-Gordon equation is solved numerically without any transformations or linearization of the nonlinear terms. The paper is structured in the following way;

Section 2 elaborates on cubic B-spline collocation method. Whereas in Section 3, Hybrid cubic B-spline basis functions are introduced. Moreover, different techniques to obtain the solution of (1)-(3) with the help of these basis functions are also explained. Preliminary vectors have been calculated in Section 4. Section 5, carries out numerical experiments to illustrate the viability and the computational efficiency of the proposed method and compares its results with some prior results. Lastly, short conclusions based on the current research are made in Section 6.

2. Description of Method

First of all, we make a partition of the domain $L_1 \leq y \leq L_2$ into a mesh which is of uniform length $h = y_{j+1} - y_j$, where $j = 0,1,2,\dots,N-1,N$, in such a way that $L_1 = y_0 < y_1 < \cdots < y_{N-1} <$

$y_N = L_2$. The numerical solution in terms of cubic B-spline technique can be written as $V_N(\mathrm{y},t) = \sum_{j=-1}^{N+1} c_j(t) B_j(y)$. For the numerical treatment of (1) using spline-based collocation technique our main concern is to obtain an approximate solution $V_N(y,t)$ to the exact solution $v(y,t)$ in the following form

$$V_N(\mathrm{y},t) = \sum_{j=-1}^{N+1} c_j(t) B_j(y) \tag{5}$$

Here $c_j(t)$ represents the quantities which are time dependent. We shall calculate them from the boundary conditions and collocation from the differential equation.

At the end points or knots the cubic B spline $B_j(\mathrm{y})$ is given by

$$B_j(y) = \frac{1}{h^3}\begin{cases} (y - y_{j-2})^3 & y \in [y_{j-2}, y_{j-1}), \\ (y - y_{j-2})^3 - 4(y - y_{j-1})^3 & y \in [y_{j-1}, y_j), \\ (y_{j+2} - y)^3 - 4(y_{j+1} - y)^3 & y \in [y_j, y_{j+1}), \\ (y_{j+2} - y)^3 & y \in [y_{j+1}, y_{j+2}), \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

Where $\{B_{-1}, B_0, B_1, \ldots, B_{N-1}, B_N, B_{N+1}\}$ are termed as the set of functions. These functions over the region $L_1 \le y \le L_2$ form a basis to abstain from the clear adjustment of the boundary valuer functions. In this method an element is covered by four cubic B-splines because each cubic B-spline covers four elements. The values of $B_j(y)$, C_N and its derivatives are tabulated in Table 1.

Here, making use of the approximate function given in (5) and Table 1, the approximate values of $V_N(y,t)$ and two of its derivatives at the knots can be calculated in the form of the time parameters c_j as follows:

$$V_j = c_{j-1} + 4c_j + c_{j+1} \tag{7}$$

$$U_j' = \frac{3}{h}(c_{j+1} - c_{j-1}),$$
$$U_j'' = \frac{6}{h^2}(c_{j-1} - 2c_j + c_{j+1}).$$

3. Numerical Scheme

We have used the following modified form of cubic B-spline basis function in the combination with collocation, to find the approximate solution of the sine-Gordon equation. Hybrid cubic B-spline technique has been put into action for the treatment of the Dirichlet boundary conditions and finally we obtain a diagonally dominant system of differential equations.

$$\begin{aligned}
\tilde{B}_0(x) &= B_0(x) + 2B_{-1}(x)\\
\tilde{B}_1(x) &= B_1(x) - B_{-1}(x)\\
\tilde{B}_j(x) &= B_j(x), \text{ for } j = 2, \dots, N-2\\
\tilde{B}_{N-1}(x) &= B_{N-1}(x) - B_{N+1}(x)\\
\tilde{B}_N(x) &= B_N(x) + 2B_{N+1}(x)
\end{aligned} \tag{8}$$

In order to solve the sine-Gordon equation given under (1), first of all it is reformed using following
transformation:

$$v_t(y,t) = u(y,t). \tag{9}$$

Now, Eq.(1) takes the form of a mathematical coupled system of equations as follows

$$\begin{aligned}
v_t &= u\\
u_t &= v_{yy} - \sin\, v
\end{aligned} \tag{10}$$

Next to solve the system in (10), we apply Hybrid cubic B-spline collocation strategy. we suppose that the approximate solution is a linear combination of the Hybrid cubic B-splines

$$V_N(y,t) = \sum_{j=0}^{N} c_j(t)\tilde{B}_j(y). \tag{11}$$

The approximate value of $V_t(y)$ can be expressed using the approximate solution (11) as below

$$V_t(y) = \sum_{j=0}^{N} \dot{c}_j(t)\tilde{B}_j(y), \tag{12}$$

Here $\dot{c}_j(t)$ represents the rate of change of $c_j(t)$ with respect to the time. With modified basis function (8) and Table 1 in (12) we would write the value of $V_t(y)$ at various knots as:

$$\begin{aligned}
V_t(y_0) &= 6\dot{c}_0, & j &= 0,\\
V_t(y_j) &= \dot{c}_{j-1} + 4\dot{c}_j + \dot{c}_{j+1}, & j &= 1,2,\dots,N-1,\\
V_t(y_N) &= 6\dot{c}_N, & j &= N.
\end{aligned} \tag{13}$$

On the coupled system (10) we apply (11) at the boundary points and enforce the boundary conditions (3).

$$V_t(y_0) = \psi_1(t), \qquad j = 0, \tag{14}$$

$$V_t(y_j) = u_j, \qquad j = 1,2,\dots,N-1,$$
$$V_t(y_N) = \dot{\psi}_2(t), \qquad j = N,$$
$$(y_0) = \dot{g}_1(t), \qquad j = 0,$$
$$v_t(y_j) = \sum_{j=0}^{N} c_j \tilde{B}_j''(x) - \sin(V_j), \qquad j = 1,2,\dots,N-1,$$
$$v_t(y_N) = \dot{g}_2(t), \qquad j = N, \qquad (15)$$

Substituting Equation (13) into Equation (14) and Equation (8), Table 1 into Equation (15) we obtain a system of the following equations:

$$6C_0 = \psi_1(t), \qquad j = 0,$$
$$c_{j-1} + 4c_j + c_{j+1} = u_j, \qquad j = 1,2,3,\dots,N-1$$
$$6C_N = \psi_2(t), \qquad \mathrm{j = N},$$
$$\dot{u} = g_1(t), \qquad \mathrm{j = 0}$$
$$v_t(y_N) = \dot{g}_2(t), \qquad j = N, \qquad (16)$$

Now, the expression (16) takes the following form

$$\begin{bmatrix} 6 & 0 & \cdots & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ & & \cdots & \cdots & \cdots \\ & & \cdots & \cdots & \cdots \\ & & 1 & 4 & 1 \\ & & & 0 & 6 \end{bmatrix} \begin{bmatrix} \dot{c}_0 \\ \dot{c}_1 \\ \cdots \\ \cdots \\ \dot{c}_{N-1} \\ \dot{c}_N \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ \cdots \\ \cdots \\ F_{N-1} \\ F_N \end{bmatrix}, \qquad (17)$$

$$\begin{bmatrix} \dot{v}_0 \\ \dot{v}_1 \\ \cdots \\ \cdots \\ \dot{v}_{N-1} \\ \dot{v}_N \end{bmatrix} = \begin{bmatrix} G_0 \\ G_1 \\ \cdots \\ \cdots \\ G_{N-1} \\ G_N \end{bmatrix}, \qquad (18)$$

Were

$$F_0 = \psi_1(t), \quad \mathrm{G}_0 = \dot{g}_1(t), \qquad F_j = v_j,$$

$$G_j = \frac{6}{h^2}\left(c_{j-1} - 2c_j + c_{j+1}\right) - \sin\left(c_{j-1} + 4c_j + c_{j+1}\right), \tag{19}$$

$$F_N = \dot{\psi}_2(t), G_N = \dot{g}_2(t).$$

After determining the values of C_N, we are now able to obtain the numerical solution at the knots that are needed. We solve expression (17) by Thomas algorithm at each time step t>0. In this way the problem of the vectors c is now resolved. Then the system obtained with the system (18) will yield (2N+ 2) first order ODEs and eventually approximate solution V_N(y,t) is obtained. To find the solution at specific time level t>0, we need the initial vectors $\mathbf{c}^0$ and $\mathbf{v}^0$.

With initial conditions (2) we obtain the following.

$$\begin{aligned} U\left(x_j,0\right) &= \psi_1(0), \quad j = 0, \\ U\left(x_j,0\right) &= \phi_1\left(x_j\right), j = 1, \dots, N-1, \\ U\left(x_j,0\right) &= \psi_2(0), \quad j = N. \end{aligned} \tag{20}$$

It is a tridiagonal system of equations with dimension $(N+1) \times (N+1)$ and can be expressed as

$$A\mathrm{c}^{\mathbf{0}} = B, \tag{21}$$

were

$$A = \begin{bmatrix} 6 & 0 & \dots & \dots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ & & \dots & \dots & \dots \\ & & \dots & \dots & \dots \\ & & 1 & 4 & 1 \\ & & & 0 & 6 \end{bmatrix}, \mathbf{c}^0 = \begin{bmatrix} \dot{c}_0 \\ \dot{c}_1 \\ \cdots \\ \dots \\ \dot{c}_{N-1} \\ \dot{c}_N \end{bmatrix}, \tag{22}$$

$$B = \begin{bmatrix} \psi_1(0) \\ \phi_1(y_1) \\ \cdots \\ \dots \\ \phi_1(y_{N-1}) \\ \psi_2(0) \end{bmatrix}. \tag{22}$$

$Where\ A$ is a tridiagonal matrix. Now, with the help of Thomas algorithm the solution under expression (20) can be easily found.
The Initial Vector $\mathbf{v}^0$ can be calculated using the initial condition

$$V_t(y,0) = \phi_2(y) \tag{23}$$

we have

$$v(y_j,0)=\phi_2(y_j), j=0,1,\dots,N-1,N, \tag{24}$$

In this way we can calculate the initial vector $\mathbf{v}^0$.

5. Numerical Experiments

Here we take four numerical examples to confirm the scheme proposed. The scheme is assessed by determining norms of errors L_2, L_∞ and order of convergence. The obtained numerical outcomes compared with some work available in the literature. L_2, L_∞, and RMS error norms are given by the following formulae:

$$L_2=\|v-V_N\|_2=\sqrt{h\sum_{j=0}^{N}\left|v_j-(V_N)_j\right|^2}$$

$$L_\infty=\|v-V_N\|_\infty=\max_j\left|v_j-(V_N)_j\right| \tag{25}$$

$$\text{RMS error }=\frac{1}{N+1}\sqrt{\sum_{j=0}^{N}\left|v_j-(V_N)_j\right|^2}$$

Problem 1. In this example, we approximate the solutions of (1) in the domain $y\in[-a,a]$ along with the initial conditions

$$v(y,0)=0, \qquad v_t(x,0)=4\text{sech}(y). \tag{26}$$

The exact solution is provided as

$$v(y,t)=\frac{2\arctan\left(t.\text{sech}(y)\right)}{1/2} \tag{27}$$

Whereas the boundary conditions are calculated from the given exact solution.

when $a=1$, with $\Delta t=.0001$and spatial step size $h=.02,.04$. We solve this problem in the domain $[-1,1]$. In Table 2, the L_2 and L_∞ errors are displayed and a comparison have made with those given in Dehghan and Shokri [16]. It can be observed that the numerical outcomes are better than those available in [16], however when $h=.04$ and $h=.02$ is taken, it produces superior approximation than the result of [16] in terms of L_2 error. Absolute errors, are reported at $t=.01$ with $h=.02$ in Table 3. Graphical representation comparing the exact and numerical solutions at $t=1$ with $h=.02$ and $\Delta t=.0001$ is portrayed in Figure 1.

As a second case, we Choose the domain [-2,2] then the numerical solutions of Problem 1 are achieved with $\Delta t=.01$ and $h=.01$. Table 4 shows the L_∞error-norms at different temporal step size and approximate results are compared with [17]. We see that obtained results are in good agreement with [17] regarding error norm L_∞. In Figure 2 the comparison of exact and approximate solutions has been depicted at $t = 1$.

Table 2: L 2 and L infinity errors for Problem 1 in the field [-1,1] at various time levels.

$time$	Proposed Method				Method in [16]	
	$h = .04$		$h = .02$		$h = .04$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	1.18×10^{-5}	2.32×10^{-5}	3.71×10^{-6}	8.20×10^{-6}	3.91×10^{-5}	5.89×10^{-6}
0.50	4.19×10^{-5}	4.11×10^{-5}	1.34×10^{-5}	1.62×10^{-5}	1.30×10^{-4}	2.01×10^{-5}
0.75	7.78×10^{-5}	1.02×10^{-4}	2.40×10^{-5}	2.54×10^{-5}	2.35×10^{-4}	3.63×10^{-5}
1.00	1.30×10^{-4}	1.64×10^{-4}	3.00×10^{-5}	4.14×10^{-5}	3.27×10^{-4}	5.07×10^{-5}

Table 3: Absolute errors with $h = .02$and $\Delta t = .001$ of Problem 1 at different time levels

y	$t = .01$	$t = .1$	$t = 1$
-0. 80	3.56×10^{-11}	4.13×10^{-8}	1.06×10^{-5}
-0. 60	1.42×10^{-11}	1.81×10^{-8}	5.88×10^{-7}
-0. 40	2.25×10^{-11}	2.58×10^{-8}	1.28×10^{-5}
0	1.04×10^{-10}	1.01×10^{-7}	3.84×10^{-5}
0. 40	2.25×10^{-11}	2.79×10^{-8}	1.36×10^{-5}
0. 60	1.42×10^{-11}	1.86×10^{-8}	5088×10^{-7}
0. 80	3.56×10^{-11}	4.10×10^{-8}	1.06×10^{-5}

Table 4: Comparison of L infinity and errors of the Problem one in the range -2 to 2 with h=.01,Delta t=.01 at various time levels.

t	Proposed Method		Method in [18]	
	L_∞	RMS	L_∞	RMS
0. 1	7.11×10^{-6}	6.25×10^{-8}	1.54×10^{-6}	7.43×10^{-6}
0. 2	2.13×10^{-5}	2.35×10^{-7}	9.25×10^{-5}	1.76×10^{-5}
0. 3	4.28×10^{-5}	6.19×10^{-7}	9.02×10^{-5}	3.60×10^{-5}
0. 4	7.27×10^{-5}	1.08×10^{-6}	1.62×10^{-4}	1.62×10^{-4}
0. 5	1.07×10^{-4}	1.49×10^{-6}	2.58×10^{-4}	1.10×10^{-4}
0. 6	1.27×10^{-4}	2.47×10^{-6}	3.73×10^{-4}	1.65×10^{-4}
0. 7	2.01×10^{-4}	4.11×10^{-6}	4.98×10^{-4}	2.29×10^{-4}
0. 8	2.35×10^{-4}	5.37×10^{-6}	6.24×10^{-4}	2.98×10^{-4}
0. 9	3.11×10^{-4}	7.27×10^{-6}	7.44×10^{-4}	3.69×10^{-4}
1	3.28×10^{-4}	9.21×10^{-6}	8.49×10^{-4}	4.37×10^{-4}

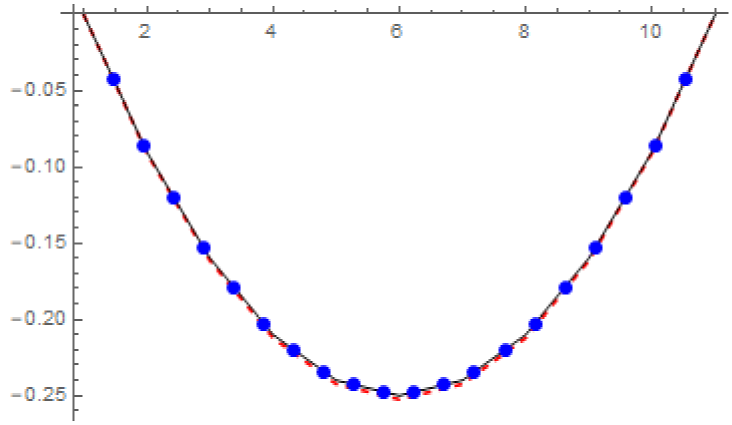

Figure 1: 2D representation of numerical and exact solution at $\Delta t = .001$, $t = 1$ with $h = .04$.

Piece wise solution of problem 1

$$
\mathrm{v}(r,t)=\begin{cases}
3.729277964441845\times 10^{(-21)}+y(3.2123439294442163+y(-0.07575695898575865 \\
+(2.338435969207705-5.5128142436886675y)\,\mathrm{y})), & \text{if } y\in[0.00,0.01) \\
-0.04373551437930211+y(2.596106266647739+y(-5.821162268441179 \\
+(15.640552539774342-26.594358597374196y)\,\mathrm{y})), & \text{if } y\in[0.01,0.02) \\
-0.665764802877835+y(6.906471656532946+y(-54.24284366661641 \\
+\left(54.925852772138455-37.43284722449995y\right)\mathrm{y})), & \text{if } y\in[0.02,0.03) \\
-5.18092471146921+y(35.79623209113032+y(-86.20039698373068 \\
+\left(75.78155670997653-47.48194751375938r\right)\mathrm{y})), & \text{if } y\in[0.03,0.04) \\
-22.60896340955324+y(55.99070591385746+y(-256.54566824258077 \\
+\left(532.77160954364062-56.81591429375769y\right)\mathrm{y})), & \text{if } y\in[0.04,0.05) \\
-45.301792773536452+y(524.80882262444684+y(-545.27507446471272 \\
+\left(123.2586954656898-24.11188297696924r\right)\mathrm{y})), & \text{if } y\in[0.05,0.06) \\
-54.99928800300447+y(358.2962741725207+y(-478.1484058496071 \\
+\left(347.32121409266165-74.87543770792986r\right)\mathrm{y})), & \text{if } y\in[0.06,0.07) \\
-425.37934484008802+y(529.67483266368854+y(-646.4591282606212 \\
+\left(409.56314258501206-75.84496178807499r\right)\mathrm{y})), & \text{if } y\in[0.07,0.08) \\
-318.28152815780413+y(758.448695935278+y(-839.4893049585749 \\
+\left(469.4384540958532-79.29562671407039r\right)\mathrm{y}, & \text{if } y\in[0.08,0.09) \\
-450.1548785760893+y(843.216031229532+y(-847.7619653336547 \\
+\left(523.48511510104854-80.9424656632466y\right)\mathrm{y}, & \text{if } y\in[0.09,0.10)
\end{cases}
$$

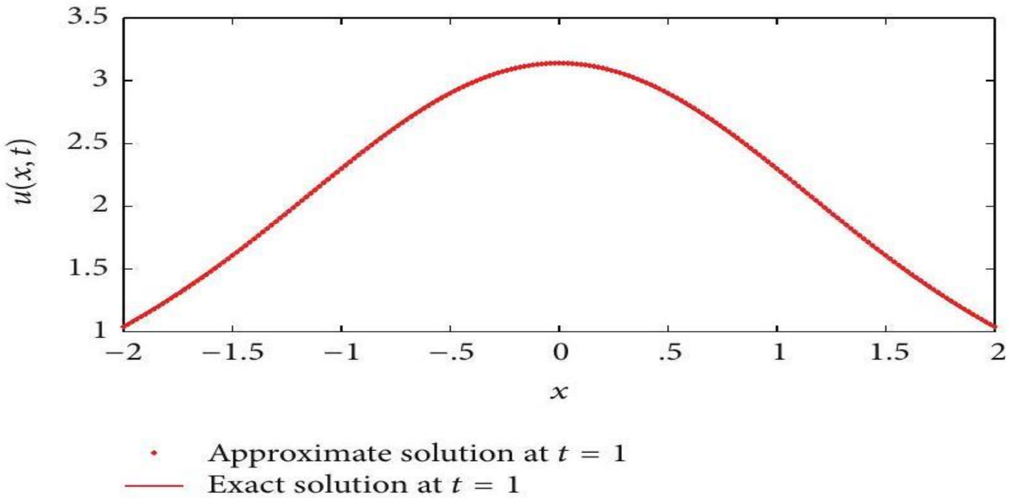

*Figure 2:*2 D graph of numerical and exact solution at $\Delta t = 0.001$, $t = 1$ and $h = .02$

Problem 2. Here we examine (1) in the interval $where\ y \in [-3,3]$ having the initial conditions

$$v(y,0) = 4\arctan\left(\exp\left(\frac{x}{\sqrt{1-c^2}}\right)\right)$$

$$v_t(y,0) = \frac{-4c\mu\exp(\mu y)}{1+\exp(2\mu y)} \qquad 28)$$

The exact solution is provided as follows

$$v(y,t) = 4\arctan(\exp(\mu(y-ct))), \qquad (29)$$

Where c represents the solitary wave velocity and $\mu = 1/\sqrt{1-c^2}$. For Problem 2, the boundary conditions are obtained from the exact solution.

Table 5 reports L_2 and L_∞ errors with various options of h using $\Delta t = 0.0001\ and\ c = 0.5$. The results have been compared with the numerical results Dehghan and Shokri [16]. Table 5 clearly indicates that the proposed method gives superior results than [16] having L_2 errors, using $h = .04$. The L_2 and L_∞ error-norms have also been calculated at $t = 1\ and\ \Delta t = 0.0001$ for different values of space step size h. Table 7 gives the order of convergence and the absolute errors at $t = 0.01$ and 0.1 are also depicted in Table 6. The graph between approximate and exact solutions at $t = 1$ is represented Figure 3.

Table 5: A comparison of L_2 and L_∞ errors at $c = 0.5$ of Problem 2 at different time levels.

t	Present Scheme				Method in [16]	
	$h = .04$		$h = .02$		$h = .04$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
.25	3.56×10^{-5}	4.46×10^{-5}	9.14×10^{-6}	1.12×10^{-6}	1.76×10^{-5}	4.95×10^{-6}
.50	8.92×10^{-5}	7.27×10^{-5}	2.13×10^{-5}	1.46×10^{-5}	4.17×10^{-5}	8.24×10^{-6}
.75	1.32×10^{-4}	1.22×10^{-4}	3.51×10^{-5}	3.27×10^{-5}	8.15×10^{-5}	1.38×10^{-5}
1.00	2.15×10^{-4}	2.05×10^{-4}	5.34×10^{-5}	5.12×10^{-5}	1.16×10^{-4}	2.26×10^{-5}

Table 6: Absolute errors with $h = 0.02$ and delta $t = 0.0001$ of Problem 2 at different time levels

y	$t = .01$	$t = .1$	$t = 1$
-2.5	5.98×10^{-10}	5.48×10^{-8}	5.14×10^{-6}
-2	8.45×10^{-10}	8.35×10^{-8}	1.15×10^{-6}
-1.5	5.33×10^{-10}	5.26×10^{-8}	9.08×10^{-8}
-1	2.40×10^{-9}	2.25×10^{-7}	2.01×10^{-5}
0	5.45×10^{-11}	5.32×10^{-8}	2.25×10^{-5}
1	2.41×10^{-9}	2.28×10^{-7}	4.43×10^{-5}
1.5	5.35×10^{-10}	5.27×10^{-8}	1.13×10^{-5}
2	8.45×10^{-10}	8.5×10^{-8}	2.12×10^{-6}
2.5	5.97×10^{-10}	5.48×10^{-8}	3.22×10^{-6}

Table 7: Error-norms with order of convergence at $t = 1$ for Problem 2.

$\bar{h}$	L_2	O.C	L_∞	O.C
.08	9.07×10^{-4}		8.39×10^{-4}	
.04	2.19×10^{-4}	2.124	2.08×10^{-4}	2.004
.02	5.59×10^{-5}	2.013	5.19×10^{-5}	2.001
.01	1.39×10^{-5}	2.007	1.29×10^{-5}	2.009
.005	3.58×10^{-6}	1.856	3.27×10^{-6}	1.785

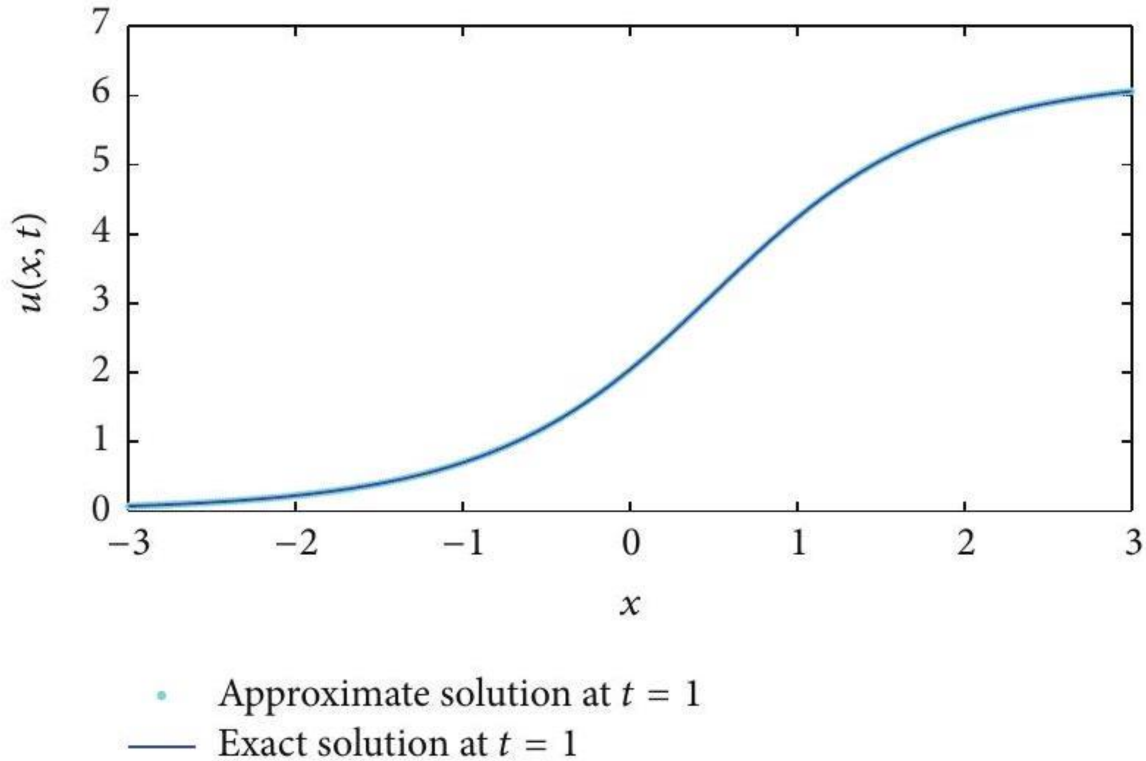

Figure 3: 2D comparison of the exact and numerical solutions at $h = .01$, $t = 1$ with $\Delta t = 0.001$.

Problem 3. Here we consider nonlinear SGE in the range $y \in [-10,10]$ with the initial conditions

$$v(y,0) = 0, \qquad v_t(y,0) = \frac{4}{\sqrt{1+c^2}}\operatorname{sech}\left(\frac{y}{\sqrt{1+c^2}}\right). \tag{30}$$

The exact solution of Problem 3 is given below

$$v(y,t) = 4\arctan\left(c^{-1}\sin(\bar{\mu}ct)\operatorname{sech}(\bar{\gamma}x)\right), \tag{31}$$

Here c denotes the solitary wave velocity and $\bar{\mu} = 1/\sqrt{1+c^2}$.

For this problem the boundary conditions in expression (3) are calculated from the exact solution. We approximate Problem 3 using $h = 0.01$, $c = 0.5$ at $\Delta t =0\ .001$. The error-norms L_2 and L_∞ are tabulated in Table 8 and compared with Uddin *et al.* [15] and Bratsos [19]. It can be noticed from Table 8; the obtained results are in good agreement. The error-norms L_2 and L_∞ for the various values of spatial step size h with $\Delta t = 0.001$ at $t = 20$ are reported in Table 9. With the help of Table 9, we compute the order of convergence. We see that the applied method posesses second order of convergence. The space-time graph of numerical solution at $\Delta t = 0.001$ for $t \le 20$ and $h = .01$ is showcased in Figure 4.

TABLE 8: L_2 and L_∞ error-norms at $\Delta t = 0.001$ of Problem 3 using $c = 0.5$ and $h = .01$at different time levels.

t	Proposed Method		Method in [19]	Method in [15]
	L_2	L_∞	L_∞	L_∞
1	7.44×10^{-6}	7.02×10^{-6}	$.9881 \times 10^{-3}$	1.474×10^{-3}
10	3.99×10^{-5}	2.22×10^{-5}	$.1629 \times 10^{-2}$	9.215×10^{-3}
20	6.4×10^{-4}	3.5×10^{-4}	$.103 \times 10^{-2}$	3.038×10^{-1}

Table 9: Error-norms and order of convergence at $t = 20$ for Problem 3.

$\bar{h}$	L_2	O.C	L_∞	O.C
0. 08	4.169×10^{-2}	-	2.302×10^{-2}	-
0. 04	1.029×10^{-2}	2.013	5.729×10^{-3}	2.012
0. 02	2.600×10^{-3}	2.010	1.430×10^{-3}	2.009
. 01	6.467×10^{-4}	2.003	3.567×10^{-4}	2.003

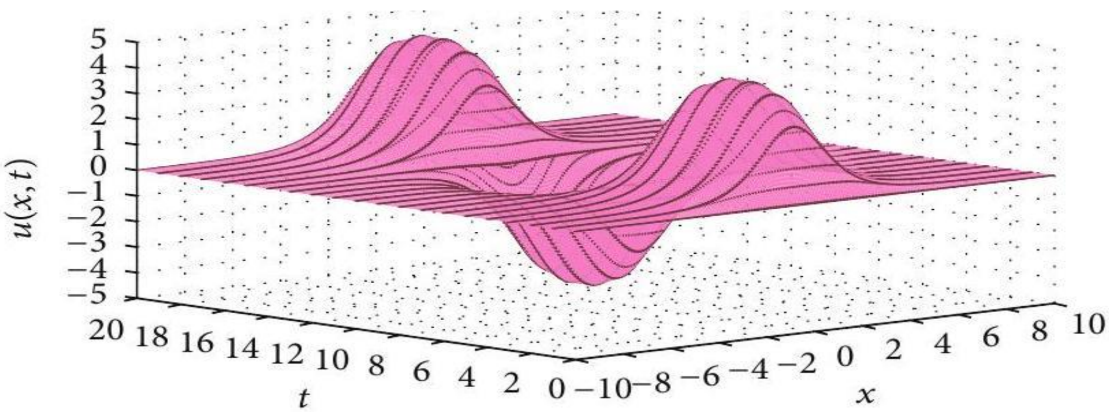

Figure 4: 3D Representation of Numerical solution up to $t = 20$ with $-10 \leq x \leq 10$.

Piece wise solution of Problem 3

$$
v(r,t)=\begin{cases}
4.6563258475452\times10^{(-21)}+y(6.325641235+y(-0.07575695898575865 \\
+(2.338435969207705-5.5128142436886675y)\,y)), & if\ y\in[0.00,0.01) \\
-0.04373551437930211+y(2.596106266647739+y(-5.821162268441179 \\
+(15.640552539774342-26.594358597374196y)\,y)), & if\ y\in[0.01,0.02) \\
-0.665764802877835+y(6.906471656532946+y(-54.24284366661641 \\
+\left(54.925852772138455-37.43284722449995y\right)y)), & if\ y\in[0.02,0.03) \\
-5.18092471146921+y(35.79623209113032+y(-86.20039698373068 \\
+\left(75.78155670997653-47.48194751375938r\right)y)), & if\ y\in[0.03,0.04) \\
-22.60896340955324+y(55.99070591385746+y(-256.54566824258077 \\
+\left(532.77160954364062-56.81591429375769y\right)y)), & if\ y\in[0.04,0.05) \\
-45.301792773536452+y(524.80882262444684+y(-545.27507446471272 \\
+\left(123.2586954656898-24.11188297696924r\right)y)), & if\ y\in[0.05,0.06) \\
-54.99928800300447+y(358.2962741725207+y(-478.1484058496071 \\
+\left(347.32121409266165-74.87543770792986r\right)y)), & if\ y\in[0.06,0.07) \\
-425.37934484008802+y(529.67483266368854+y(-646.4591282606212 \\
+\left(409.56314258501206-75.84496178807499r\right)y)), & if\ y\in[0.07,0.08) \\
-318.28152815780413+y(758.448695935278+y(-839.4893049585749 \\
+\left(469.4384540958532-79.29562671407039r\right)y, & if\ y\in[0.08,0.09) \\
-450.1548785760893+y(843.216031229532+y(-847.7619653336547 \\
+\left(523.48511510104854-80.9424656632466y\right)y, & if\ y\in[0.09,0.10)
\end{cases}
$$

Problem 4. Finally, we consider (1) in the range $y \in [-20,20]$ with the initial conditions

$$v(y,0) = 4\arctan\left(c.\sinh\left(\frac{y}{\sqrt{1-c^2}}\right)\right), v_t(y,0) = 0. \tag{32}$$

The exact solution is provided below as

$$v(y,t) = \arctan.4\big(c.\sinh(\mu x)\,\mathrm{sech}(\mu ct)\big), \tag{33}$$

Here c represents the solitary wave velocity and $\mu = 1/\sqrt{1-c^2}$. For this Problem the boundary conditions in expression (3) are calculated from the exact solution.

We solve Problem 4 at $\Delta t = 0.001$ with $c = 0.5$ and $h = 0.01$. The numerical outcomes are depicted in Table 10. The computed results are compared with [15] and [11] in terms of L_∞ error-

errors and it can be observed that they are in good agreement. The error norms $L-2$ and $L-$ infinity for various values of spatial step size h at $\Delta t = .001$ and $t = 20$ are reported in Table 11. With the help of these error-norms the order of convergence of the applied technique has been calculated. The graph in Figure 5 shows the numerical solution at $\Delta t = 0.001$, for $t \leq 20$ and spatial step size $h = 0.01$.

Table 10: L_2 and L_∞ errors of Problem 4 using $c = 0.5$ and at different time levels.

t	Proposed Method		Method in [11]	Method in [15]
	$h = 0.01$		$h = 0.01$	M = 200
	L_2	L_∞	L_∞	L_∞
2	2.228×10^{-5}	1.524×10^{-5}	1.135×10^{-3}	1.365×10^{-3}
10	8.458×10^{-5}	5.152×10^{-5}	1.856×10^{-3}	3.025×10^{-3}
20	1.452×10^{-4}	9.253×10^{-5}	2.247×10^{-3}	1.412×10^{-2}

Table 11: L-infinity and L-2 norms with order of convergence for Problem 4 using $t = 15$.

$\bar{h}$	$L-2$	O.C	L-infinity	O.C
.08	1.094×10^{-2}	-------	6.025×10^{-3}	-------
.04	2.740×10^{-3}	1.997	1.509×10^{-3}	1.997
.02	6.851×10^{-4}	1.999	3.775×10^{-4}	1.999
.01	1.713×10^{-4}	1.999	9.438×10^{-5}	1.999

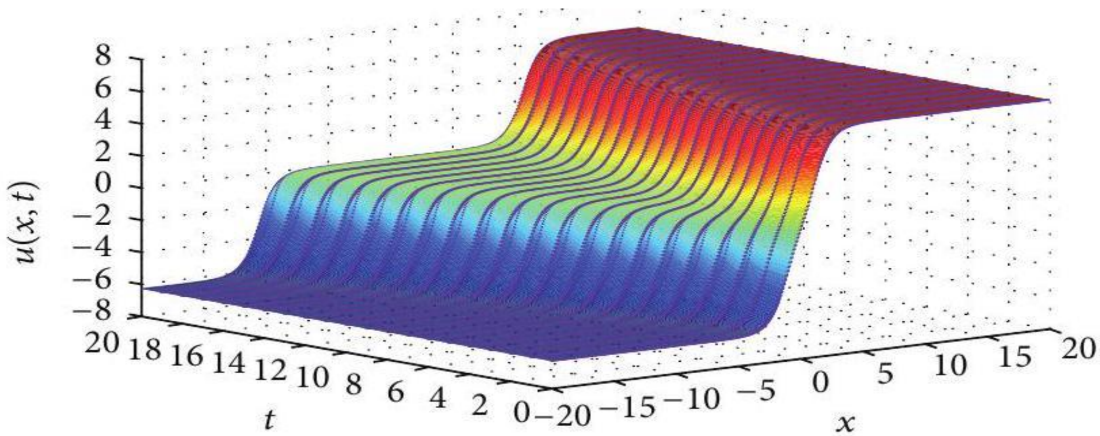

Figure 5: 3D representation of Numerical solution up to $t = 20$ with $-20 \leq x \leq 20$.

6. Conclusions

This paper discusses the numerical approximation of non-linear Sine-Gordon Equation (SGE) via B-spline collocation approach making use of the Dirichlet boundary condition by converting it in the form of a system of equations. The application of a mix of the Hybrid cubic B-spline basis functions to space discretization and the time integration scheme of SSP-RK54 is effective in generating results that are accurate and reliable. It is noteworthy that the numerical solutions are obtained without using any transformation or the linearization process. Four numerical experiments are performed to confirm the effectiveness and validity of the suggested approach with the results compared to the existing studies. The results show that the technique fares well as compared to the literature. Moreover, the calculated orders of convergence in Tables 7, 9, and 11 prove that the method is second-order accurate. The obtained numerical results prove that the proposed technique is highly reliable and applicable technique and approximates the solutions very well.

References

[1] M. Shaheen, M. Abbas, F. A. Abdullah, and Y. S. Hamed, "A new numerical technique for the solution of time-fractional nonlinear Klein-Gordon equation involving Atangana-Baleanu derivative using cubic B-spline functions," *Open Phys.*, vol. 23, no. 1, Jan. 2025, doi: 10.1515/PHYS-2025-0131/PDF.

[2] G. Ben-Yu, P. J. Pascual, M. J. Rodriguez, and L. Vázquez, "Numerical solution of the sine-Gordon equation," *Appl. Math. Comput.*, vol. 18, no. 1, pp. 1–14, 1986, doi: 10.1016/0096-3003(86)90025-1.

[3] M. Amin, M. Abbas, M. K. Iqbal, and D. Baleanu, "Numerical Treatment of Time-Fractional Klein–Gordon Equation Using Redefined Extended Cubic B-Spline Functions," *Front. Phys.*, vol. 8, no. September, pp. 1–13, 2020, doi: 10.3389/fphy.2020.00288.

[4] R. M. Ganji, H. Jafari, M. Kgarose, and A. Mohammadi, "Numerical solutions of time-fractional Klein–Gordon equations by clique polynomials," *Alex. Eng. J.*, vol. 60, no. 5, pp. 4563–4571, Oct. 2021, doi: 10.1016/j.aej.2021.03.026.

[5] A. Mohebbi and M. Dehghan, "High order solution of one-dimensional sine-Gordon equation using compact finite difference and DIRKN method," *Math Comput Model*, vol. 51, no. 5–6, pp. 537–549, Mar. 2010, doi: 10.1016/j.mcm.2009.11.015.

[6] C. Lu, W. Huang, and J. Qiu, "An Adaptive Moving Mesh Finite Element Solution of the Regularized Long Wave Equation," *J. Sci. Comput.*, vol. 74, no. 1, pp. 122–144, 2018, doi: 10.1007/s10915-017-0427-6.

[7] K. Djidjeli, W. G. Price, and E. H. Twizell, "Numerical solutions of a damped sine-Gordon equation in TWo space variables," *J Eng Math*, vol. 24, no. 4, pp. 347–369, Jul. 1995, doi: 10.1007/bf00042761.

[8] N. A. Kudryashov, "Exact solutions of the generalized Kuramoto-Sivashinsky equation," *Phys. Lett. A*, vol. 147, no. 5–6, pp. 287–291, Jul. 1990, doi: 10.1016/0375-9601(90)90449-X.

[9] B. Batiha, M. S. M. Noorani, and I. Hashim, "Approximate analytical solution of the coupled sine-Gordon equation using the variational iteration method," *Phys. Scr.*, vol. 76, no. 5, pp. 445–448, Nov. 2007, doi: 10.1088/0031-8949/76/5/007.

[10] R. C. Mittal and R. Bhatia, "Numerical Solution of Nonlinear Sine-Gordon Equation by Modified Cubic B-Spline Collocation Method," *Int. J. Partial Differ. Equations*, vol. 2014, pp. 1–8, Aug. 2014, doi: 10.1155/2014/343497.

[11] A. G. Bratsos, "A fourth order numerical scheme for the one-dimensional sine-Gordon equation," *Int. J. Comput. Math.*, vol. 85, no. 7, pp. 1083–1095, Jul. 2008, doi: 10.1080/00207160701473939.

[12] A. Shokri and M. Dehghan, "A meshless method using the radial basis functions for

numerical solution of the regularized long wave equation," *Numer. Methods Partial Differ. Equ.*, vol. 26, no. 4, pp. 807–825, 2010, doi: 10.1002/num.20457.

[13] J. Rashidinia and R. Mohammadi, "Tension spline solution of non-linear sine-Gordon," *Numer Algorithms*, vol. 56, no. 1, pp. 129–142, Jan. 2011, doi: 10.1007/s11075-010-9377-x.

[14] Z. W. Jiang and R. H. Wang, "Numerical solution of sine-Gordon equation using high accuracy multiquadric quasi-interpolation," *Appl Math Comput*, vol. 218, no. 15, pp. 7711–7716, Apr. 2012, doi: 10.1016/j.amc.2011.12.095.

[15] M. Uddin, S. Haq, and Siraj-ul-Islam, "A mesh-free numerical method for solution of the family of Kuramoto–Sivashinsky equations," *Appl. Math. Comput.*, vol. 212, no. 2, pp. 458–469, Jun. 2009, doi: 10.1016/j.amc.2009.02.037.

[16] M. Dehghan and A. Shokri, "A numerical method for one-dimensional nonlinear sine-gordon equation using collocation and radial basis functions," *Numer. Methods Partial Differ. Equ.*, vol. 24, no. 2, pp. 687–698, 2008, doi: 10.1002/NUM.20289.

[17] M. Dehghan and A. Shokri, "Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions," *J. Comput. Appl. Math.*, vol. 230, no. 2, pp. 400–410, Aug. 2009, doi: 10.1016/j.cam.2008.12.011.

[18] University of Texas, "No TitleΕΛΕΝΗ," *Afghanisan Physiogr. Map*, vol. 31, no. 2, pp. 1–106, 2009.

[19] A. G. Bratsos, "A fourth order numerical scheme for the one-dimensional sine-Gordon equation," *Int. J. Comput. Math.*, vol. 85, no. 7, pp. 1083–1095, Jul. 2008, doi: 10.1080/00207160701473939.