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NUMERICAL VALUATION OF THE BLACK–SCHOLES EQUATION VIA THE CRANK NICOLSON FINITE DIFFERENCE SCHEME

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Article Info

Abstract

In this research, we investigate a popular Black-Scholes model in terms of a full financial sector system, where choices are valued and assessed. The Crank–Nicolson finite difference technique (FDT) is used to offer a numerical solution for the coupled partial differential equation (PDE). The accuracy and stability features of the suggested method, as well as its ease of use, are the basis for its growth. The usefulness of the approach for the given situation is assessed by numerical experiments, and the validity of the Crank–Nicolson technique in estimating option pricing is confirmed by analyzing the calculated results. The results show that the suggested approach is useful, computationally effective, and appropriate for resolving financial PDEs, particularly when simplicity and convenience of use are crucial.

Keywords:
BSM (Black-Scholes model), Crank Nicolson FDM, Option pricing, PDE, Numerical methods, Financial Mathematics.

1. Introduction:

The valuation of financial derivatives, and in particular options, is one of the most important topics of computational finance. In 1973, Fischer Black and Myron Scholes presented the renowned Black–Schales model [1] that provided a mathematical framework for option pricing (European) using the method of partial differential equations (PDE).

Since it was introduced, the Black–Schale's equation has been fundamental in financial mathematics, and has inspired a lot of research in both analytical and numerical solution methods. The Black-Scholes model is based on the following assumptions: The underlying asset is assumed to have constant interest rate and volatility, and the price of the underlying asset moves in a random way. The value of an option under these assumptions is a solution of the parabolic partial differential equation that does not usually have closed-form analytical solutions for all types of options and boundary conditions. In this way, there are many practical applications which widely use numerical methods [2, 3].

The finite difference methods (FDM) are among the most popular numerical techniques due to their simplicity, stability and effectiveness in approximating solutions to Pes [11, 12]. One of the most popular finite difference methods to solve a parabolic PDE is the Crank–Nicolson method. This approach has the good features of both explicit and implicit schemes, second-order accurate in both time and space and numerically stable [7, 11]. Because they make the Crank–Nicolson method very popular numerically in option pricing problem in context of the Black – Schale's equation.

This technique is a time and asset price discretization, and transforms the continuous PDE into a set of algebraic equations which can be solved iteratively. In the last few years, there have been several researchers working on numerical approaches and predictor–corrector methods to financial PDFs. The linearly implicit predictor–corrector (PICR) schemes for reaction–diffusion equations, developed by Khaliq and Voss [4] influenced on the design of stable numerical solution to the financial model equations. Hilbert [9] continued to study the predictor–corrector methods for option pricing. Similarly, Dura and Mosneagu [6] presented on numerical solutions of the Black-Schales equation; and emphasized the need of efficient discretization approaches in computational finance. Stochastic calculus and stochastic processes have a strong connection to the mathematical basis of the Black–Schales model. The modelling of the dynamics of asset prices is based on the theory of stochastic processes, as have been developed by researchers such as David Nualart [5].

Besides, the classical Black-Schols framework has been extended to investigate the stochastic volatility and long-memory model which can better describe the real markets [8]. Thanks to the modern developments in numerical analysis, there are improvements in Crank–Nicolson-based technique as well. The modified Crank–Nicolson methods have been proven to be effective for solving linear PDFs by Sharma [14] and variational quantum Crank–Nicolson methods have been studied for solving initial value problems in recent years by Guzman-Cajica [13].

The studies show the ongoing applicability and versatility of the Crank–Nicolson framework in scientific computing and financial mathematics. This paper is intended to look into the implementation of Crank–Nicholson finite difference method for the Black-Scholes equation. The major part of the paper is devoted to the construction of the numerical scheme, the realization of the boundary and initial conditions, the stability and accuracy analysis of the numerical scheme used for pricing European options. This method is stable and will provide an accurate estimation of the price of options, which can be done in an efficient manner with the Crank–Nicolson technique, and therefore is suitable for computational finance applications to the real world.

2.Black-Scholes PDE (Partial Differential Equation)

The pricing function of a European call a given option, represented as $F(t, y)$, can be used to determine the ideal hedging portfolio whenever the wealth process $X(t)$ is seen as a semi-martingale. Consequently, the wealth process can be written as:

In following Ito's rule is utilized

$$\frac{\partial}{\partial t}F\big[t,s[t]\big]+\gamma S[t]\frac{\partial}{\partial y}F\big[t,s[t]\big]+\frac{1}{2}\frac{\partial^2}{\partial y^2}F\big[t,s[t]\big]\,\sigma^2\,S^2\,[t]-\gamma.F\big[t,s[t]\big]=0\ ,$$

$$\frac{\partial F}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial F^2}{\partial S^2}+(\gamma-q)S\frac{\partial F}{\partial S}-\gamma.F=0$$

Above is known as so-called Black-Schole-Merton PDE for which boundary is: $(t, s[t]) = (s[t] - k)^+$.

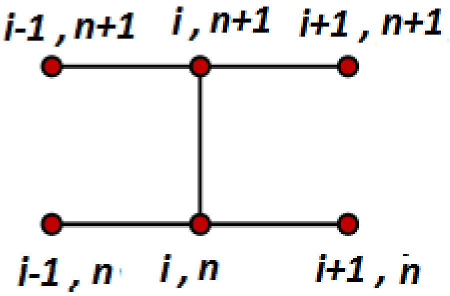
There are so many variations connected to the deviations that can be connected with the basic variable. Based on the understanding of the condition, the specific subsidiary which will be acquired depends on the limit conditions used.

Crank Nicolson Standard

Thus, the beginning value of the option can be found by applying the numerical technique given in the equation and solving it backward in time using the payout conditions at the maturity time T. Since the necessary values are already known from the previous time step, the scheme is explicit, meaning that no system of algebraic equations needs to be solved.

There is a convergent subsequence of the sequence if the following statement is true:

$$0<\Delta t<\frac{1}{\sigma^2(N-1)+\frac{1}{2}r}.$$



Fully implicit technique needs to pull in focal points – genuine steadiness , anyway, it could bring about superfluous computational many-sided quality to value a choice.

One other better method to value alternative, however, is the Crank-Nicolson technique (C-N), possibly a cross section in between the unequivocal approach and the completely certain approach.

If we assume the C-N thought with respect to the BS, then we obtain the set of corresponding lattice condition:

$$\left.\frac{\partial F}{\partial S}\right|_{(S_i,t_{n+1})} = \left(\frac{F_{i+1}^{n+1}-V_{i-1}^{n+1}}{2h} + \frac{V_{i+1}^{n}-F_{i-1}^{n}}{2h}\right) + Oh^2,$$

$$\left.\frac{\partial^2 F}{\partial S^2}\right|_{(S_i,t_{n+1})} = \left(\frac{F_{i+1}^{n+1}+F_{i-1}^{n+1}-2F_{i}^{n+1}}{h^2} + \frac{F_{i+1}^{n}+F_{i-1}^{n}-2F_{i}^{n}}{h^2}\right) Oh^2,$$

$$\left.\frac{\partial F}{\partial t}\right|_{(S_i,t_{n+1})} = \frac{F_{i}^{n+1}-F_{i}^{n}}{\Delta t} + Oh^2,$$

Once more, we examine the Black-Scholes equation and demonstrate an implicit numerical scheme using it.

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial F^2}{\partial S^2} + (\gamma - q)S\frac{\partial F}{\partial S} - r.F = 0,$$

i.e.

$$\frac{F_i^{n+1} - F_i^n}{\Delta t} + \frac{1}{2}\sigma^2 (ih)^2 \frac{1}{2}\left(\frac{F_{i+1}^{n+1} + F_{i-1}^{n+1} - 2F_i^{n+1}}{h^2} + \frac{F_{i+1}^{n} + F_{i-1}^{n} - 2F_i^{n}}{h^2}\right) +$$

$$(\gamma - q)ih\frac{1}{2}\left(\frac{F_{i+1}^{n+1}-F_{i-1}^{n+1}}{2h} + \frac{F_{i+1}^{n}-F_{i-1}^{n}}{2h}\right) - \frac{1}{2}\gamma(F_i^n + F_i^{n+1}) = 0, \qquad 4.10$$

we rewrite as:

$$(1+\sigma^2 i^2\Delta t + r\Delta t)F_i^n - \left(\tfrac{1}{2}\sigma^2 i^2 + \tfrac{1}{2}ri\right)\Delta t F_{i+1}^n - \left(\tfrac{1}{2}\sigma^2 i^2 + \tfrac{1}{2}r\right)\Delta t V_{i-1}^n - rF_i^{n+1} = 0. \qquad 4.11$$

Let

$$\alpha_i = \tfrac{1}{4}(\sigma^2 i^2 - \gamma i)\Delta t \qquad i = 2,3,\ldots\ldots,N-2,$$
$$\beta_i = -\tfrac{1}{2}(\sigma^2 i^2 + \gamma)\Delta t \qquad i = 0,1,2,3,\ldots\ldots,N-1,$$
$$\gamma_i = \tfrac{1}{4}(\sigma^2 i^2 + \gamma i)\Delta t \qquad i = 1,2,3,\ldots\ldots,N-1,$$

$$CC = \begin{bmatrix} 1-\beta_1 & -\gamma_1 & 0 & \ldots & 0 \\ -\alpha_1 & 1-\beta_2 & -\gamma_2 & & \vdots \\ 0 & \ddots & d_3 & \ddots & 0 \\ \vdots & \ddots & & \ddots & -\gamma_{N-1} \\ 0 & \ldots & \ldots & -\alpha_{N-2} & 1-\beta_N \end{bmatrix}, DD = \begin{bmatrix} 1+\beta_1 & \gamma_1 & 0 & \ldots & 0 \\ \alpha_1 & 1+\beta_2 & \gamma_2 & & \vdots \\ 0 & \ddots & d_3 & \ddots & 0 \\ \vdots & \ddots & & \ddots & \gamma_{N-1} \\ 0 & \ldots & & \alpha_{N-2} & 1+\beta_N \end{bmatrix},$$

$$K = \begin{bmatrix} \alpha_0 . V_0^{n+1} \\ 0 \\ 0 \\ \vdots \\ \gamma_N . V_N^{n+1} \end{bmatrix},$$

then

$$CC.F_i^{n+1} = DD.F_i^n + K,$$
$$F_i^{n+1} = (CC)^{-1}(DD.F_i^n + K). \qquad 4.13$$

By calculating F_i^n , there appeared F_i^{n+1} for time t. Here two expressions F_{i-1}^n and F_{i+1}^n are given at time t. Hence, for every time step, the number of equations and the number of unknowns are both equal. The pay-off condition at time T will not be enough, two more boundary conditions are required in order to be able to solve the system. Usually we obtain these additional equations using the boundary conditions i.e. by having the value of the option for all the time steps at the stock price S=0 and S=S_max : F_0^n and F_M^n ∀n . Then we've got the system under a matrix of the form

This is a Crank Nicolson scheme, as we have to solve a set of algebraic equations which are very time consuming. However, this is a scheme that is unconditionally convergent. Hence this type of methods very interesting due to their stability in spite of the computational efficiency will be loss.

Nonlinear partial differential equation Black-Scholes Finite difference Technique is used, Black-Schols (Crank Nicolson Technique) and (Crank Nicolson Technique)

Exact solution by Black-Scholes option pricing formula

The European Black-Scholes option pricing formula provides a precise solution for the European call option problem, especially when the interest rate and volatility stay constant.

$$F^{c}(t,y) = y\psi(D_1) - ke^{-r(T-t)}\psi(D_2).$$

the standardized normal random variable, Where $\Phi(.)$ is the cumulative distribution function for given by

$$\psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du,$$

$$D_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\frac{y}{k} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right],$$

$$D_2 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\frac{y}{k} + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right].$$

1. Problem:

The current stock price for an option is 32 $. The existence price for the maturity period of 6 months is $20, interest rate risk free is 20% per anum, volatility of the stock is 80% per anum. Let the option is a European call option then find value of it.

Data:

S=32, r=0.2, k=20, T=0.5, $\sigma = 0.8$

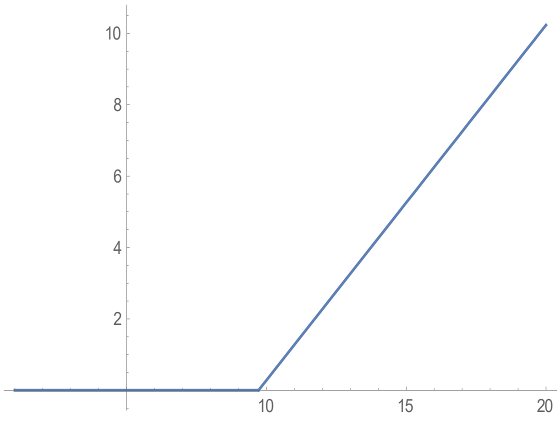
Comparison, Numerical Results and their Graphs

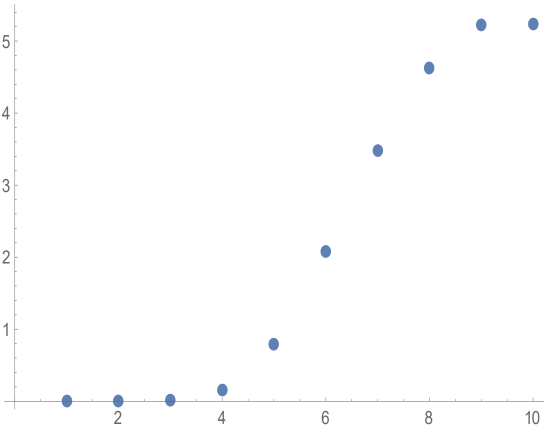
S=32, r=0.2, k=20, T=0.5, $\sigma = 0.8$

Call Option Value		P=10, Q=100	P=100, Q=1000	P=200, Q=10000
Exact		**0.149334843518360**	**0.149334843518360**	**0.149334843518360**
Crank-Nicolson Scheme		**0.097892170154668**	**0.103918247238172**	**0.104960638142454**

Exact Solution Crank Nicolson

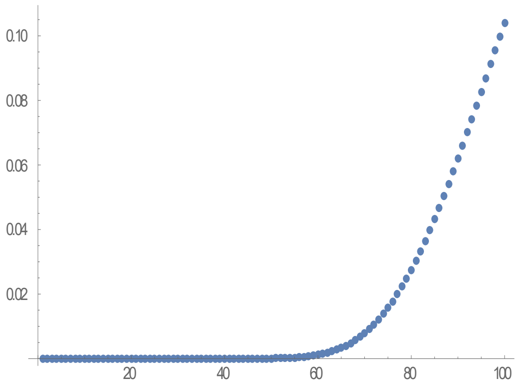
Iterations: q= 10, P=100, 100 iterations

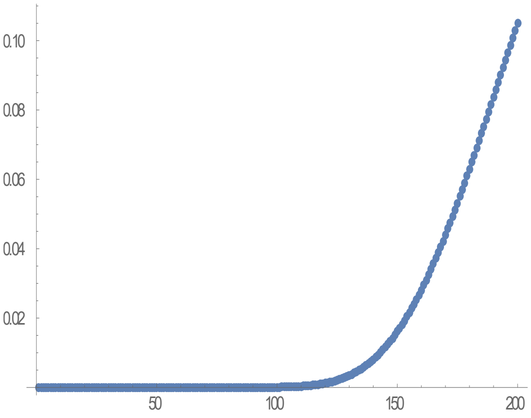




Solution of crank Nicloson Solution of Crank Nicoloson

Iterations: q= 100, p=100, 100th iteration Iterations: , q=100,p=100, 1000 iteration





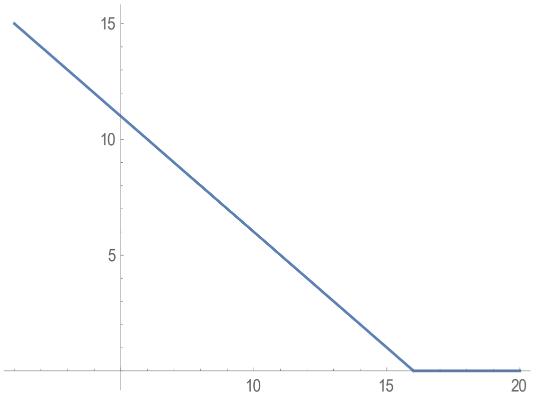
2. **Problem:**
3. The current stock price for an option is 32 $. The existence price for the maturity period of 6 months is $20, interest rate risk free is 20% per anum, volatility of the stock is 80% per anum. Let the option is a European put option then find value of it.

S=32, r=0.2, k=20, T=0.5, $\boldsymbol{\sigma = 0.8}$

Put Option Value	Q=10, P=100	Q=100, P=1000	Q=200,P=10000
Exact	**1.90243396**	**1.90243396**	**1.90243396**
Crank-Nicolson Scheme	**1.90210783**	**1.90213965**	**1.90212515**

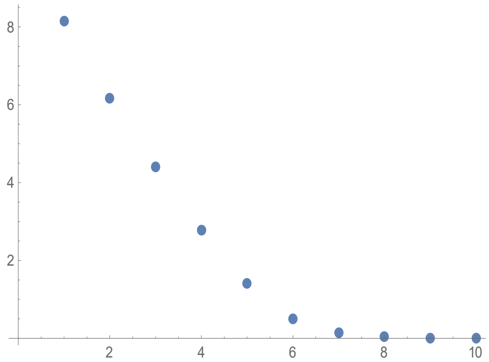
Exact Solution Crank Nicolson Solution

Iterations: q= 10 p=100 , 100 iterations



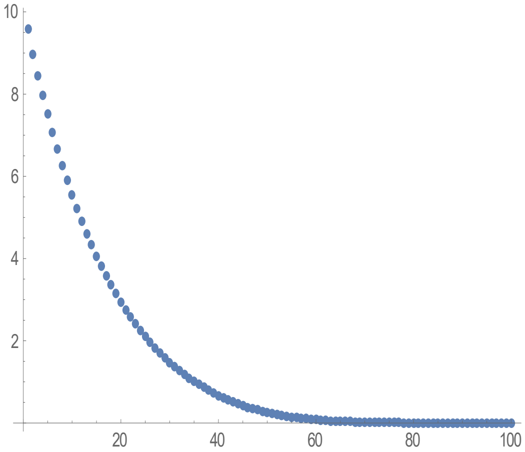
Solution of crank Nicoloson

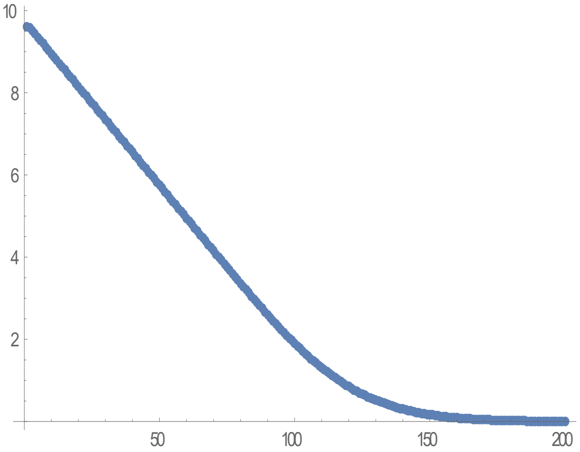
Iterations: q= 10 p=100 100th iterations



Solution of crank NiSolosons

Iterations: q= 10 p=100 , 1000 iterations





Conclusion:

In this paper, the option pricing Black-Scholes equation was discussed using finite difference Crank-Nicolson method. Although the Black–Schales model is a simple and useful mathematical model for pricing European options, sometimes it is hard to get an analytical solution of this pricing model for practical and complex financial problems. So the use of numerical techniques is a crucial part of computational finance.

The Crank–Nicolson scheme was used due to its stability, accuracy and efficiency in the parabolic partial differential equations. The Crank–Nicolson method enjoys the best of both worlds: It is numerically stable, second order accurate in time, and second order accurate in space. This was because Black-Scholes was a continuous equation, and needed to have a discrete approximation, which was solved through an iterative numerical computation.

The acquired results indicate that the Crank-Nicolson algorithm can provide a good, reliable approximation of option prices. This method will converge smoothly and minimize the numerical errors as compared to many of the traditional finite difference methods. Also, it is flexible with regard to different boundaries and initial conditions that can be associated with financial derivative pricing problems.

Furthermore, the subject of the study contains information about the significance of numerical analysis in the current financial mathematics. B/lack and Scholes [1] and Hull [2] present a theory and computation on option pricing in finite difference methods, whereas Higham [3] and Strikwerda [11] and Wilmott et al. [12] present details about option pricing in finite difference methods. Also it is worth mentioning that the sophisticated numerical techniques in order to solve financial PDFs have been gradually more important in the recent developments of the numerical schemes, the modified Crank – Nicolson approaches [13,14].

In conclusion, Crank–Nicolson method is found to be an efficient and applicable numerical method for the solution of the Black Schales equation. It is also possible to use the method in more complex financial models, like multidimensional derivative pricing problems, barrier-options, American-options, and stochastic volatility models. Possible future extensions of the present work are further improvements in the computation efficiency, minimization of discretization errors, and also the inclusion of the recent advances in the fields of numerical and quantum computations such as the simulation of large-scale financial systems.

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