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DISCRETIZATION OF LAPLACIAN OPERATOR IN POLAR COORDINATE SYSTEM, USING CRANK-NICOLSON'S (CN) SCHEME AND STABILITY ANALYSIS

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Article Info

Abstract

Laplacian operator plays a vital role for describing and solving many mathematical models. Finite difference Scheme of Laplacian operator has been carried out by various researchers using Euler's scheme but unfortunately the scheme is stable for very small ratio of k/h^2. However, Crank-Nicolson's scheme has been proved stable for all values of k/h^2. In this contribution discretization of Laplacian operator in polar coordinate system will be carried by using Crank-Nicolson's Scheme. Stability analysis has been carried out for the scheme and results are compared with previous research analysis. The obtained isotropic discretized Laplacian operator on the 5-points stencil on the polar grid system shows stable and accuracy as compared to previously obtained discretization using explicit finite difference scheme. The result obtained on various sizes of polar mesh system shows that the error of the isotropic discrete Laplacian decreases rapidly with each step time step of the computation scheme. The derived isotropic discretized Laplacian scheme on the polar net system is highly useful for employing and studying various computational models. The acquired discretization of the polar Laplacian scheme on the circular annular plane is potential candidate for significant results for the CDS (Cell Dynamic Simulations) model used for advanced materials and image processing used for computational studies. Centre of the circle and its neighborhood have been avoided in this study due to the limitations of Laplacian operator in the polar coordinate system.

Keywords: *Laplacian operator, Crank-Nicolson's scheme, Discretization, Polar coordinate system.*

INTRODUCTION

In this contribution, discretization of Laplacian operator is carried out by finite difference scheme. The discretization of the Laplacian operator has been carried out with the help of Crank-Nicolson's (CN) scheme which is the most stable method available in the literature. In this work, 5-point stencil Laplacian operator has been discretized in polar coordinate system using Crank-Nicolson's method, I choose polar coordinate system in order to get more isotropic results. The Laplacian operator was transformed from Cartesian coordinate system to polar coordinate system, after that discretization process was carried out in polar mesh system. For this process first and second order partial derivatives with respect to radius r and measure of angle Ɵ were approximated using Crank-Nicolson's method. These derivatives are substituted in the Laplacian operator and an appropriate course of action is followed for discretization to obtain average value of neighboring points. To carry out computations, the reflexive boundary conditions were applied on the radial coordinate whereas, periodic boundary conditions were applied on angular coordinates. The results are found by obtained isotropic discretized Laplacian operator an implicit method and by an explicit method FD scheme with corresponding errors comparing with exact values/ analytical values. This scheme gives stable and accurate results as compared with previously obtained discretization using explicit finite difference scheme. The derived isotropic discrete Laplacian scheme on the polar net system is highly useful for employing and studying various computational models. The obtained discretization of the Laplacian in polar coordinate system on the circular annular plane is potential agent for significant results for the CDS (Cell Dynamic Simulations) model used for advanced materials and image processing used for computational studies. Centre of the circle and its neighbourhood have been avoided in this study due to the limitations of Laplacian operator in the polar coordinate system. Also, the corresponding errors are calculated for comparison with explicit FD scheme.

Laplacian operator plays a significant role to describe and evaluate various Mathematical and computational models (Geertjan Huiskamp, 1991). The Laplacian is extremely important in mechanics, electromagnetics wave theory, quantum mechanics , material science and ecology and appears in Laplace's equation (Weisstein, no date). As compared other developing explicit schemes especially for convergence and stability have less truncation error and are convergent without condition and stable. John Crank and Phyllis Nicolson in the year 1947 proposed a scheme by introducing a fictitious time level at (j+1/2) (Campin *et al.*, 2004).This method is unconditionally stable and has obvious property of parallelism (Baolin and Wenzhi, 1994).This is a highly stable and accurate method for discretization. Discretization is the important tool to obtain the approximate values for Differential Operator, the derivatives are approximated by using FD scheme, CN scheme etc. (Watts, 2013). The best of my knowledge no one worked on discretization of Laplacian operator in polar coordinates system, using Crank-Nicolson method which is most stable method available in the literature.

METHODOLOGY

In many applications and computational models' Laplacian operator is essentially required for isotropic and accurate computation of the physical quantities under consideration. In this work, Laplacian operator in continues time domain will be discretized in the circular annular pore using polar mesh system. The Crank-Nicolson's scheme will be used for the discretization of the Laplacian operator instead of explicit finite difference scheme to obtain more stable scheme. The Laplacian operator can be expressed in Cartesian coordinates system as below,

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \psi_{xx} + \psi_{yy} \tag{1}$$

Changing Cartesian Coordinates in polar coordinates we have,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

By differentiating above parametric equations to get partial derivatives as,

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Using Chain Rule, $\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial\psi}{\partial y}.\frac{\partial y}{\partial r} = \frac{\partial\psi}{\partial x}.\cos\theta + \frac{\partial\psi}{\partial y}.\sin\theta$

$$\frac{\partial\psi}{\partial r} = \cos\theta\frac{\partial\psi}{\partial x} + \sin\theta\frac{\partial\psi}{\partial y} \tag{2}$$

Differentiating (2) with respect to r to get 2nd derivative of function ψ with respect to r,

$$\frac{\partial^2\psi}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}.\frac{\partial\psi}{\partial x} + \sin\theta\frac{\partial}{\partial r}.\frac{\partial u}{\partial y} = \cos\theta\left(\frac{\partial}{\partial x}.\frac{\partial\psi}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}.\frac{\partial\psi}{\partial x}.\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}.\frac{\partial\psi}{\partial y}.\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}.\frac{\partial\psi}{\partial y}.\frac{\partial y}{\partial r}\right)$$

$$\frac{\partial^2\psi}{\partial^2 r^2} = \cos^2\theta\frac{\partial^2\psi}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\partial y} + \sin^2\theta\frac{\partial^2\psi}{\partial y^2} \tag{3}$$

Again, using Chain Rule, $\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial x}.\frac{\partial x}{\partial\theta} + \frac{\partial\psi}{\partial y}.\frac{\partial y}{\partial\theta} = \frac{\partial\psi}{\partial x}(-r\sin\theta) + \frac{\partial\psi}{\partial y}.r\cos\theta$

$$\frac{\partial\psi}{\partial\theta} = -r\sin\theta\frac{\partial\psi}{\partial x}\cos\theta + r\cos\theta\frac{\partial\psi}{\partial y} \tag{4}$$

Differentiating (4) to get 2nd derivative of function ψ with respect to θ

$$\frac{\partial^2\psi}{\partial\theta^2} = -r\cos\theta\frac{\partial\psi}{\partial x} - r\sin\theta\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial x} - r\sin\theta\frac{\partial\psi}{\partial y} + r\cos\theta\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial y}$$

$$\frac{\partial^2\psi}{\partial\theta^2} = -r\cos\theta\frac{\partial\psi}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}.\frac{\partial\psi}{\partial x}.\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}.\frac{\partial\psi}{\partial x}.\frac{\partial y}{\partial\theta}\right) - r\sin\theta\frac{\partial\psi}{\partial y} + r\cos\theta\left(\frac{\partial}{\partial x}.\frac{\partial\psi}{\partial y}.\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}.\frac{\partial\psi}{\partial y}.\frac{\partial y}{\partial\theta}\right)$$

$$\frac{\partial^2\psi}{\partial\theta^2} = -r\cos\theta\frac{\partial\psi}{\partial x} - r\sin\theta\left(\frac{\partial^2\psi}{\partial x^2}(-r\sin\theta) + \frac{\partial^2\psi}{\partial x\partial y}r\cos\theta\right)$$

$$\frac{\partial^2\psi}{\partial\theta^2}=-r\sin\theta\frac{\partial\psi}{\partial y}+r\cos s\theta\left(\frac{\partial^2\psi}{\partial x\partial y}(-r\sin\theta)+\frac{\partial^2\psi}{\partial y^2}r\cos\theta\right)$$

$$\frac{\partial^2\psi}{\partial\theta^2}=-r\left(\cos\theta\frac{\partial\psi}{\partial x}+\sin\theta\frac{\partial\psi}{\partial y}\right)+r^2\left(\sin^2\theta\frac{\partial^2\psi}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\partial y}+\cos^2\theta\frac{\partial^2\psi}{\partial y^2}\right)$$

Using (2), here is the value of $\frac{\partial\psi}{\partial r}$

$$\frac{\partial^2\psi}{\partial\theta^2}=-r\frac{\partial\psi}{\partial r}+r^2\left(\sin^2\theta\frac{\partial^2\psi}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\partial y}+\cos^2\theta\frac{\partial^2\psi}{\partial y^2}\right)$$

Dividing both sides by r^2

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=-\frac{1}{r}\frac{\partial\psi}{\partial r}+\left(\sin^2\theta\frac{\partial^2\psi}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\partial y}+\cos^2\theta\frac{\partial^2\psi}{\partial y^2}\right) \tag{5}$$

Adding (3) and (5)

$$\frac{\partial^2\psi}{\partial^2 r^2}+\frac{1}{r^2}\frac{\partial\psi}{\partial\theta^2}=\cos^2\theta\frac{\partial^2\psi}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\partial y}+\sin^2\theta\frac{\partial^2\psi}{\partial y^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\sin^2\theta\frac{\partial^2\psi}{\partial x^2}$$
$$-2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\partial y}+\cos^2\theta\frac{\partial^2\psi}{\partial y^2}$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial x^2}\left(\cos^2\theta+\sin^2\theta\right)+\frac{\partial^2\psi}{\partial y^2}\left(\cos^2\theta+\sin^2\theta\right)$$

As we know that

$$\cos^2\theta+\sin^2\theta=1$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\ OR\ \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

And finally, it can be written in polar coordinates form as below,

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \tag{6}$$

The discretization of the Laplacian operator in polar coordinate system, using finite differences schemes have been obtained in the circular annular domain which is given below (Soomro, 2016),

$$\psi_{(i,j)}=\frac{r^2.\Delta r^2.\Delta\theta^2}{2(\Delta r^2+\Delta\theta^2.r^2)}\left[\frac{\psi_{(i+1,j)}+\psi_{(i-1,j)}}{\Delta r^2}+\frac{\psi_{(i+1,j)}-\psi_{(i-1,j)}}{2r.\Delta r}+\frac{\psi_{(i,j+1)}+\psi_{(i,j-1)}}{r^2.\Delta\theta^2}\right] \tag{7}$$

$$\nabla^2\psi = \frac{r^2.\Delta r^2.\Delta\theta^2}{2(\Delta r^2 + \Delta\theta^2.r^2)}\left[\frac{\psi_{(i+1,j)} + \psi_{(i-1,j)}}{\Delta r^2} + \frac{\psi_{(i+1,j)} - \psi_{(i-1,j)}}{2r.\Delta r} + \frac{\psi_{(i,j+1)} + \psi_{(i,j-1)}}{r^2.\Delta\theta^2}\right] - \psi_{(i,j)} \tag{8}$$

In CN method 1st derivatives with respect to r can be represented by following relation shown below,

$$\frac{\partial\psi}{\partial r} = \frac{1}{2}\left(\frac{\psi(r_{i+1,j}^{n+1}) - \psi(r_{i-1,j}^{n+1})}{2\Delta r} + \frac{\psi(r_{i+1,j}^{n}) - \psi(r_{i-1,j}^{n})}{2\Delta r}\right) \tag{9}$$

And the 2nd derivatives with respect to r and θ respectively can be represented as shown below,

$$\frac{\partial^2\psi}{\partial r^2}(r_{i,j}^{n+\frac{1}{2}}) = \frac{1}{2}\left(\frac{\psi(r_{i+1,j}^{n+1}) - 2\psi(r_{i,j}^{n+1}) + \psi(r_{i-1,j}^{n+1})}{(\Delta r)^2} + \frac{\psi(r_{i+1,j}^{n}) - 2\psi(r_{i,j}^{n}) + \psi(r_{i-1,j}^{n})}{(\Delta r)^2}\right) \tag{10}$$

$$\frac{\partial^2\psi}{\partial\theta^2}(\theta_{i,j}^{n+\frac{1}{2}}) = \frac{1}{2}\left(\frac{\psi(\theta_{i,j+1}^{n+1}) - 2\psi(\theta_{i,j}^{n+1}) + \psi(\theta_{i,j-1}^{n+1})}{(\Delta\theta)^2} + \frac{\psi(\theta_{i,j+1}^{n}) - 2\psi(\theta_{i,j}^{n}) + \psi(\theta_{i,j-1}^{n})}{(\Delta\theta)^2}\right). \tag{11}$$

Boundary conditions are applied on angular. These derivatives are substituted in the Laplacian operator and an appropriate course of action is followed for discretization. The reflexive boundary conditions are applied on the radial coordinate and periodic coordinates.

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

By Crank-Nicolson's method,

$$\psi = \frac{\psi_{(i,j)}^{n} + \psi_{(i,j)}^{n+1}}{2}$$

$$0 = \frac{1}{2}\left[\frac{\psi_{(i+1,j)}^{n+1} - 2\psi_{(i,j)}^{n+1} + \psi_{(i-1,j)}^{n+1}}{\Delta r^2} + \frac{\psi_{(i+1,j)}^{n} - 2\psi_{(i,j)}^{n} + \psi_{(i-1,j)}^{n}}{\Delta r^2}\right] + \frac{1}{4r}\left[\frac{\psi_{(i+1,j)}^{n+1} - \psi_{(i-1,j)}^{n+1}}{\Delta r} + \frac{\psi_{(i+1,j)}^{n} - \psi_{(i-1,j)}^{n}}{\Delta r}\right]$$
$$+ \frac{1}{2r^2}\left[\frac{\psi_{(i,j+1)}^{n+1} - 2\psi_{(i,j)}^{n+1} + \psi_{(i,j-1)}^{n+1}}{(\Delta\theta)^2} + \frac{\psi_{(i,j+1)}^{n} - 2\psi_{(i,j)}^{n} + \psi_{(i,j-1)}^{n}}{(\Delta\theta)^2}\right]$$

$$or\ \frac{\psi_{(i,j)}^{n+1}}{(\Delta r)^2} + \frac{\psi_{(i,j)}^{n}}{(\Delta r)^2} + \frac{\psi_{(i,j)}^{n+1}}{r^2(\Delta\theta)^2} + \frac{\psi_{(i,j)}^{n}}{r^2(\Delta\theta)^2} = \frac{1}{2}\left[\frac{\psi_{(i+1,j)}^{n+1} + \psi_{(i-1,j)}^{n+1}}{(\Delta r)^2} + \frac{\psi_{(i+1,j)}^{n} + \psi_{(i-1,j)}^{n}}{(\Delta r)^2}\right] + \frac{1}{4r}\left[\frac{\psi_{(i+1,j)}^{n+1} - \psi_{(i-1,j)}^{n+1}}{\Delta r}\right.$$
$$\left. + \frac{\psi_{(i+1,j)}^{n} - \psi_{(i-1,j)}^{n}}{\Delta r}\right] + \frac{1}{2r^2}\left[\frac{\psi_{(i,j+1)}^{n+1} + \psi_{(i,j-1)}^{n+1}}{(\Delta\theta)^2} + \frac{\psi_{(i,j+1)}^{n} + \psi_{(i,j-1)}^{n}}{(\Delta\theta)^2}\right]$$

$$or[\psi_{(i,j)}^{n+1} + \psi_{(i,j)}^{n}]\frac{[r^2(\Delta\theta)^2 + (\Delta r)^2]}{r^2(\Delta r)^2(\Delta\theta)^2} = \left[\frac{\psi_{(i+1,j)}^{n+1} + \psi_{(i-1,j)}^{n+1} + \psi_{(i+1,j)}^{n} + \psi_{(i-1,j)}^{n}}{2(\Delta r)^2}\right] + \left[\frac{\psi_{(i+1,j)}^{n+1} - \psi_{(i-1,j)}^{n+1} + \psi_{(i+1,j)}^{n} - \psi_{(i-1,j)}^{n}}{4r\Delta r}\right]$$
$$+ \left[\frac{\psi_{(i,j+1)}^{n+1} + \psi_{(i,j-1)}^{n+1} + \psi_{(i,j+1)}^{n} + \psi_{(i,j-1)}^{n}}{2r^2(\Delta\theta)^2}\right]$$

Finally th e Discretiza tion of Laplacian in polar coordinate s system obtained on 5 - points stencils using Crank Nicolson Method is obtained as bellow;

$$\psi^{n+1}_{(i,j)} = \frac{r^2.\Delta r^2.\Delta\theta^2}{2\left(r^2.\Delta\theta^2 + \Delta r^2\right)}\left[\frac{\psi^{n+1}_{(i,+1,j)} + \psi^{n+1}_{(i-1,j)} + \psi^{n}_{(i+1,j)} + \psi^{n}_{(i-1,j)}}{\Delta r^2} + \frac{\psi^{n+1}_{(i+1,j)} - \psi^{n+1}_{(i-1,j)} + \psi^{n}_{(i+1,j)} - \psi^{n}_{(i-1,j)}}{2r.\Delta r} + \frac{\psi^{n+1}_{(i,j+1)} + \psi^{n+1}_{(i,j-1)} + \psi^{n}_{(i,j+1)} + \psi^{n}_{(i,j-1)}}{r^2.\Delta\theta^2}\right] - \psi^{n}_{(i,j)} \qquad (12)$$

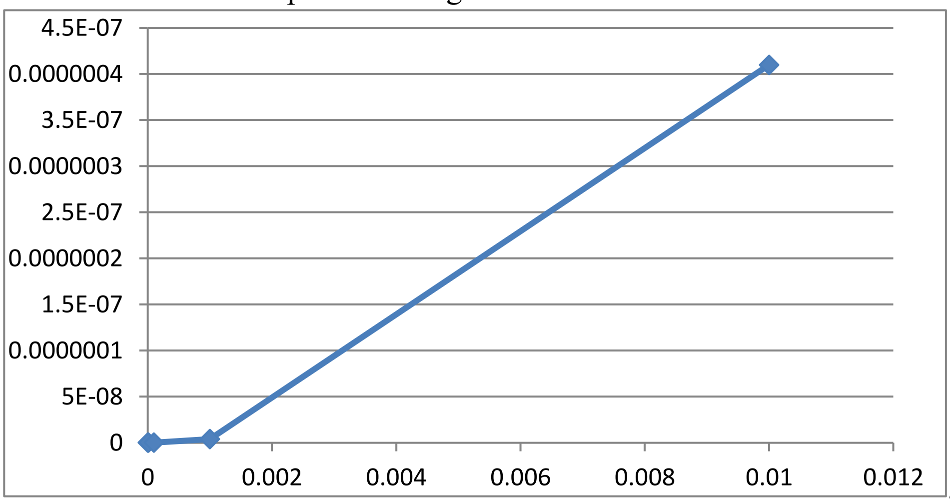
$$\nabla^2\psi = \frac{r^2.\Delta r^2.\Delta\theta^2}{2\left(r^2.\Delta\theta^2 + \Delta r^2\right)}\left[\frac{\psi^{n+1}_{(i,+1,j)} + \psi^{n+1}_{(i-1,j)} + \psi^{n}_{(i+1,j)} + \psi^{n}_{(i-1,j)}}{\Delta r^2} + \frac{\psi^{n+1}_{(i+1,j)} - \psi^{n+1}_{(i-1,j)} + \psi^{n}_{(i+1,j)} - \psi^{n}_{(i-1,j)}}{2r.\Delta r} + \frac{\psi^{n+1}_{(i,j+1)} + \psi^{n+1}_{(i,j-1)} + \psi^{n}_{(i,j+1)} + \psi^{n}_{(i,j-1)}}{r^2.\Delta\theta^2}\right] - \psi^{n}_{(i,j)} - \psi^{n+1}_{(i,j)} \qquad (13)$$

STABILITY ANALYS: Stability requires that the error not grows from iteration to iteration of the time-stepping scheme. We would either like the error to remain constant or ideally, we would like to reduce the error. Mathematically error approaches to zero as we make grid finer. To check stability of the scheme, the graph of obtained data is shown below:

Table 3.1: Table of values showing decrement in Δr against change in error

Change in Δr	Change in error
0.000001	4.1×10^{-15}
0.00001	4.1×10^{-13}
0.0001	4.1×10^{-11}
0.001	4.1×10^{-09}
0.01	4.1×10^{-07}

Graph of error against decrement in *rad*



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Change in error

Change in Δr Figure 3.1 Graph shows the decrement in error as Δr is decreased.

RESULTS

The obtained isotropic discrete Laplacian by CN method verified by hand- calculations. We used the scientific calculator without fixing it. The various discrete values for the radian and angular are chosen for the hand calculations. I choose the value of Δr as 0.1 units and Δθ as 0.0174532925199 radians (equivalent to one degree) which is changed in radian measure. I select the initial values of r and θ as 0.2units and 0.03490658504 radians (equivalent to two degrees) respectively. I use equation (7) which is explicit method known as finite difference (FD) method to find the values of function ψ as shown in the table **4.1** and (12) consisting on Crank Nicolson's scheme which is 5-points implicit method to find the values of same function ψ to compare with an explicit Finite Difference (FD) scheme as shown in same table **4.1**. I also calculate the analytic/exact values of the given function in order to compare with numerical values found by FD scheme and CN scheme. I also calculate the corresponding errors for comparison. I formed a large graph on paper whose part for one point, a 5-point stencil

Computational Molecule is shown in the figure bellow;

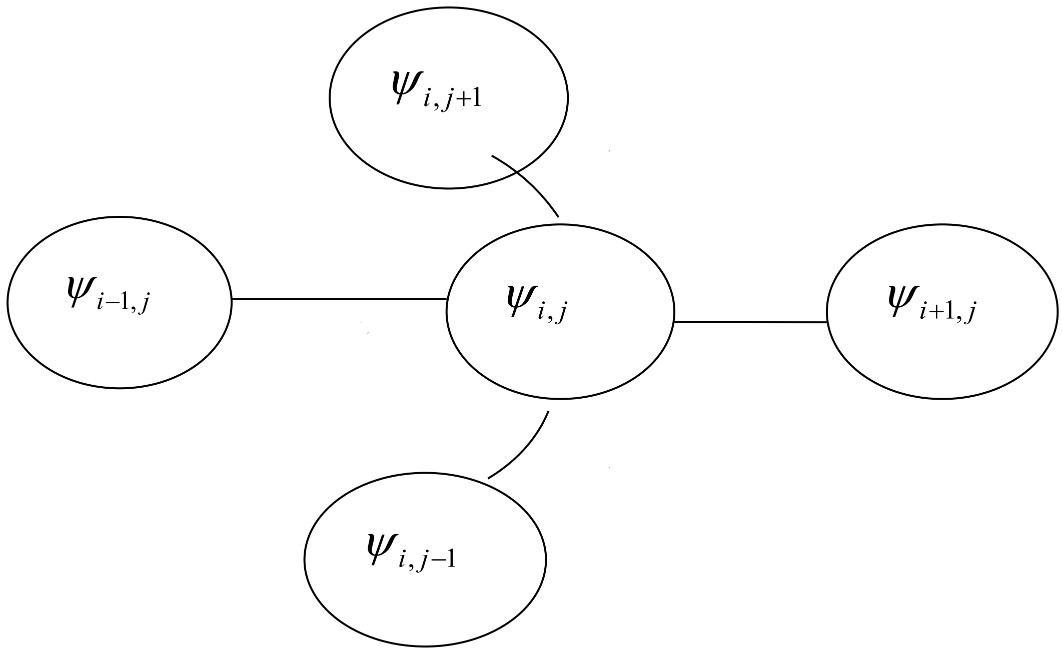

Figure: 4.1: 5-Point stencil in polar mesh system.

Example: $\psi_{(i,j)} = r^2 \cos\theta$

For the discretization scheme following notations are adopted.

$$r_i = i(\Delta r), \qquad i = 1,2,3,\ldots,n$$
$$\theta_j = j(\Delta\theta) \qquad j = 1,2,3,\ldots,m$$

In the polar mesh system following values of the constants are chosen;

$\Delta r = 0.1 \ units$

$\Delta\theta = \dfrac{\pi}{180} \ radians$

ψ Represents the order parameter in the system

Table 4.1: Obtained computational values of Explicit and Implicit scheme with corresponding errors

r_i cm	θ_j Radian	Analytic /exact values of $\psi_{(i,j)}$	Numerical values of $\psi_{(i,j)}$ by FD Scheme	Errors for FD Scheme	Numerical values of $\psi_{(i,j)}$ by Crank Nicolson Scheme	Errors for CN method
0.2	0.03490658504	0.03997563308	0.03999354 0.04001212398 0.4003044327	0.00001790692 0.00003648992 0.00005481019	0.04004880972 0.04005767302	0.00007317664116 0.0000820397115
0.2	0.05235987756	0.03994518139	0.03996341128 0.03998148004 0.03996343497	0.00001822989 0.00003629865 0.00001825358	0.04001796008 0.04003665108	0.0000727786925 0.00009146969193
0.2	0.06981317008	0.03990256201	0.03992077 0.03993898681 0.03995713279	0.00001820799 0.0000364248 0.00005457078	0.03997535276 0.04002659389	0.00007279075427 0.000124031877
0.2	0.0872664626	0.03984778792	0.03986597 0.03988416614 0.03990236396	0.00001818208 0.00003637822 0.00005457604	0.0399205325 0.03993531623	0.00007274458378 0.00008752830786
0.2	0.1047197551	0.03978087581	0.03979903077 0.03981719014 0.03982857657	0.00001815496 0.00003631433 0.00004770076	0.03984673595 0.0398649266	0.00006586014189 0.0000840507886
0.3	0.034906585	0.08994517443	0.08998612141 0.09030326773 0.09007491972	0.00004094698 0.0003580933 0.00012974529	0.09011604571 0.09029442658	0.0001708712824 0.0003492521476
0.3	0.05235987756	0.08987665813	0.08991761152 0.08995863918 0.0901373708	0.00004095339 0.00008198105 0.00026071267	0.08969256821 0.09026681463	0.00018408992 0.0003901564989
0.3	0.06981317008	0.08978076452	0.08982167609 0.08986276746 0.08999268713	0.00004091157 0.00008200294 0.00021192261	0.08994462159 0.09009501445	0.0001638570701 0.0003142499259
0.3	0.0872664626	0.089658752283	0.08969837803 0.08973922322 0.08977993601	0.000039625747 0.000080470937 0.000121183727	0.08982108017 0.08990653752	0.00016355734 0.0002490146853
0.3	0.1047197551	0.08950697058	0.089547719 0.08958856453 0.08962945847	0.00004074842 0.00008159395 0.00012248789	0.08967030201 0.08971171362	0.0001633314268 0.0002047430367
0.4	0.034906585	0.1599025323	0.1599752422 0.1600479843 0.1601214475	0.0000727099 0.000145452 0.0002189152	0.1601936037 0.1602665621	0.0002910713983 0.0003640298267
0.4	0.05235987756	0.1597807256	0.15985338	0.0000726544 0.0001453413	0.1600714866 0.1601446997	0.0002907610208 0.000922780639

			0.1599260669 0.1599989126	0.000218187		
0.4	0.06981317008	0.159610248	0.159682825 0.1597554343 0.1598280763	0.000072577 0.0001451863 0.0002178283	0.1599006856 0.1599736508	0.0002904376115 0.1599736508
0.4	0.0872664626	0.1593911517	0.159463629 0.1595361387 0.1596086806	0.0000724773 0.000144987 0.0002175289	0.1596811903 0.1599404041	0.0002900385839 0.0005492523719
0.4	0.1047197551	0.1591235033	0.1591958589 0.1592682468 0.1597111768	0.0000723556 0.0001447435 0.0005876735	0.1594130547 0.1595855276	0.000289551417 0.0004620242677
0.5	0.034906585	0.2498477068	0.2499610069 0.2500744508 0.2501924379	0.0001132992 0.000226744 0.0003447311	0.250303492 0.2503799372	0.000455785196 0.0005322304442
0.5	0.05235987756	0.2496573837	0.2497705975 0.2498888331 0.2499216865	0.0001132138 0.0002314494 0.0002643028	0.2501104151 0.2502220691	0.0004530314425 0.0009065977301
0.5	0.06981317008	0.2493910126	0.2495041062 0.2496172567 0.2497329178	0.0001130936 0.0002262441 0.0003419052	0.2498460695 0.2503316612	0.0004550568814 0.0009406485601
0.5	0.0872664626	0.2490486745	0.2491616292 0.2492746098 0.2502127683	0.0001129547 0.0002259353 0.0011640938	0.2495072996 0.24986885	0.0008292809623 0.0008201755025
0.5	0.1047197551	0.2486304738	0.248743222 0.2488560281 0.24947908262	0.0001127482 0.0002255543 0.00084860882	0.2490818403 0.249605813	0.0004513665011 0.000975339215
0.6	0.03490658504	0.3597806977	0.3599433091 0.3601059907 0.3603252246	0.0001626121 0.0003252937 0.0005445276	0.3604879077 0.3606016619	0.0007072100465 0.0008209641536
0.6	0.05235987756	0.3595066325	0.35966912 0.3598316781 0.3599949077	0.0001624875 0.0003250456 0.0004882752	0.3601574658 0.3603874462	0.0003767680866 0.0008808136958
0.6	0.06981317008	0.3591230581	0.359285373 0.3594489237 0.3596903858	0.0001623149 0.0003258656 0.0005673277	0.3598539366 0.3599659059	0.0007308784545 0.0008428477913
0.6	0.0872664626	0.3586300913	0.3587945427 0.3591164517 0.3591183053	0.0001644514 0.0004863604 0.000488214	0.3592781071 0.3593622909	0.000648015812 0.0007321996429
0.6	0.1047197551	0.3580278823	0.3581897015 0.3583539207 o.3585937243	0.0001618192 0.0003260384 0.000565842	0.358757947 0.3589516524	0.0007300647263 0.0009237701229

The designed discretization scheme for Laplacian in polar coordinates system is consistent and convergent because Δr is proportional to the error of the scheme. Following table4.2 shows the chosen decreased values of Δr against corresponding error values of discretized values of Laplacian in polar coordinates system.

Table 4.2: Decreasing values of Δr and corresponding error values of explicit and implicit scheme

Radius(r_i) cm	Δr	Angle measure (θ_j) radian	Analytic /exact values of $\psi_{(i,j)}$	Numerical values of $\psi_{(i,j)}$ by FD Scheme	Corresponding Errors for F D Scheme	Values of $\psi_{(i,j)}$ By implicit scheme	Corresponding Errors for C N Scheme
0.3	0.1	0.05235987756	0.08987665813	0.08991761319 0.08995858669 0.08999958029	0.00004095506 0.00008192856 0.00012292216	0.0900405537 9 0.0900070388 1	0.00016389566 0.00013038068
0.03	0.01	0.05235987756	0.0008987665813	0.0008991761319 0.0008995858669 0.0008999958029	0.0000004095506 0.0000008192856 0.0000012292216	0.0009004055 379 0.0009000703 881	0.00000163895 66 0.00000130380 68
0.003	0.00 1	0.05235987756	0.00000898766581 3	0.000008991761319 0.000008995858669 0.000008999958029	0.0000000040955 06 0.0000000081928 56 0.0000000122922 16	0.0000090040 55379 0.0000090007 03881	0.00000001638 9566 0.00000001303 8068
0.4	0.1	0.05235987756	0.1597807256	0.15985338 0.1599260669 0.1599989126	0.0000726544 0.0001453413 0.000218187	0.1600714866 0.1601446997	0.00029076102 08 0.00092278063 9
0.04	0.01	0.05235987756	0.001597807256	0.0015985338 0.001599260669 0.001599989126	0.000000726544 0.000001453413 0.00000218187	0.0016007148 66 0.0016014469 97	0.00000290761 0208 0.00000922780 639
0.004	0.00 1	0.05235987756	0.00001597807256	0.000015985338 0.00001599260669 0.00001599989126	0.0000000072654 4 0.0000000145341 3 0.0000000218187	0.0000160071 4866 0.0000160144 6997	0.00000002907 610208 0.00000009227 80639
0.5	0.1	0.05235987756	0.2496573837	0.2497705975 0.2498888331 0.2499216865	0.0001132138 0.0002314494 0.0002643028	0.2501104151 0.2502220691	0.00045303144 25 0.00090659773 01
0.05	0.01	0.05235987756	0.002496573837	0.002497705975 0.002498888331 0.002499216865	0.000001132138 0.000002314494 0.000002643028	0.0025011041 51 0.0025022206 91	0.00000453031 4425 0.00000906597 7301
0.005	.001	0.05235987756	0.00002496573837	0.00002497705975 0.00002498888331 0.00002499216865	0.0000000113213 8 0.0000000231449 4 0.0000000264302 8	0.0000250110 4151 0.0000250222 0691	00000.0004530 314425 0.00000009065 977301
0.6	0.1	0.05235987756	0.3595066325	0.35966912 0.3598316781 0.3599949077	0.000162487 0.000325045 0.000488275	0.3601574658 0.3603874462	0.00037676808 66 0.00088081369 58
0.06	.01	0.05235987756	0.003595066325	0.0035966912 0.003598316781 0.003599949077	0.000001624875 0.000003250456 0.000004882752	0.0036015746 58 0.0036038744 62	0.00000376768 0866 0.00000880813 6958

0.006	.001	0.05235987756	0.00003595066325	0.000035966912 0.00003598316781 0.00003599949077	0.00000001624875 0.00000003250456 0.00000004882752	0.00003601574658 0.00003603874462	0.00000003767680866 0.00000008808136958

Conclusions:

In this contribution, discretization of Laplacian operator in polar coordinates system is carried out using Crank-Nicolson Method. Polar mesh system is most important to achieve isotropy in any potential distribution system. CN method is most important method due to characteristic of unconditional stability. In this work, results are tested against exact results showed stability and convergence. This study obtains stable and accurate FTCS scheme for Laplacian in polar coordinate system using crank-Nicolson's scheme. This work consists on 5-points stencil one point in the left, other in the right, third point is above and fourth point is below the required point in the polar mesh system. Results of this scheme are compared with previous work (Discretization of Laplacian operator using finite difference scheme an explicit scheme) and with analytical values (exact values) and the corresponding errors are also calculated. Polar coordinate system is chosen due to isotropic results as compared with rectangular coordinate system. The stability of this scheme has been verified through hand calculations. The results through this scheme are obtained hand calculations using a simple scientific calculator. In hand calculation, reflexive boundary conditions were applied radial and periodic boundary conditions on angular coordinate. The Crank-Nicolson's method and polar mesh systems are both potential candidate for accuracy and isotropy in the computational models. The potential uses of this achieved discretization of Laplacian are Cell Dynamics simulations method for studying advanced ma.

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